Linear Classification

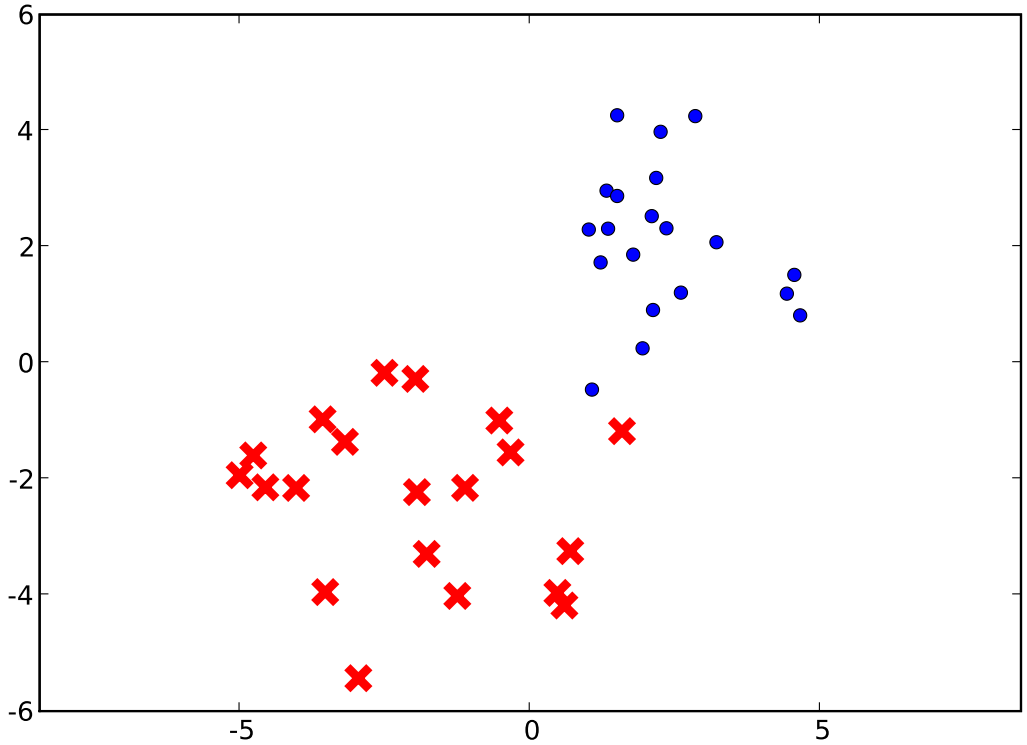
MLTA -2013

SAU, New Delhi

Dec. 15, 2013

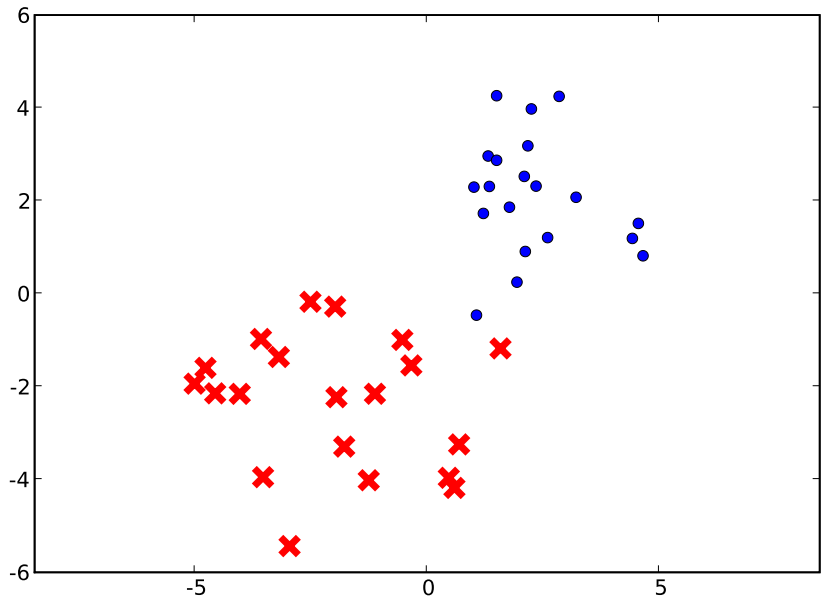
Basic Setup

- We want to separate the X's and the O's
- Today, we will see how to solve this (seemingly) simple task mathematically



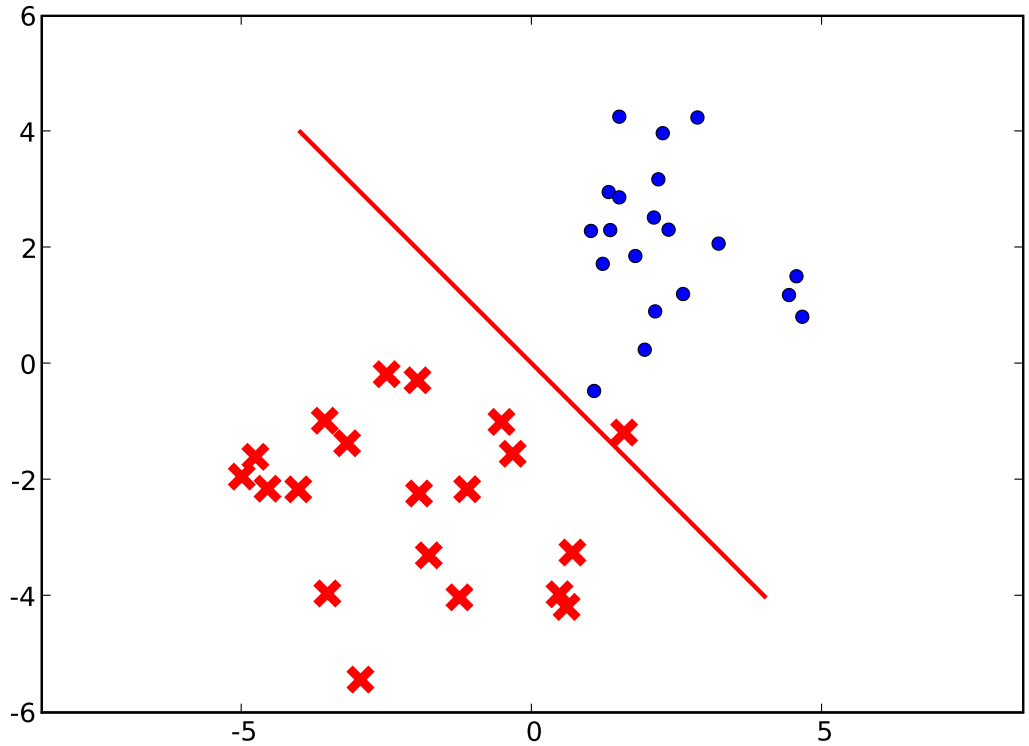
Basic Vocabulary

- We'll call each thing that we are classifying a *point*
- Each point is described by a set of numbers that we call *features*
  - It is your job to decide on the features. You need to pick features that help you do your job.
  - For instance, if you are looking for edges, you'll probably want to look at the response of different derivative filters.
- In the graph below, every point is described by two features.
- The point at index i in the dataset will be described as $\mathbf{x}_i$



Separating Points Linearly

- In building mathematical models for classifying, we are going to focus on dividing these points with a straight line.
- This is called linear classification



Expressing Linear Separation Mathematically

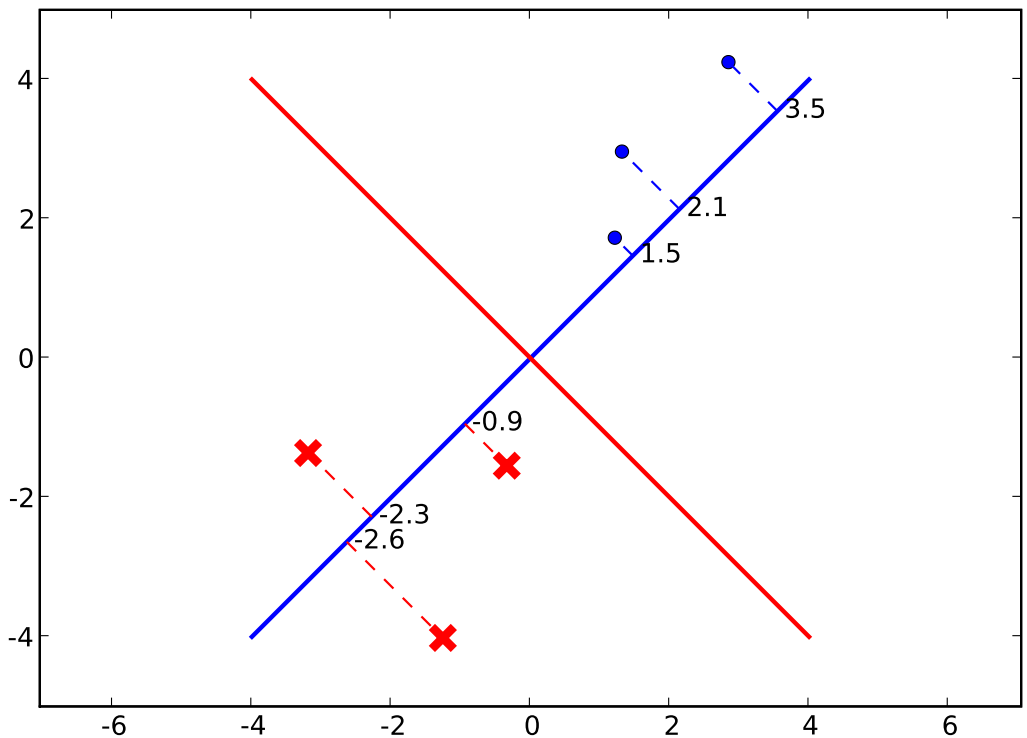
- Given a single point $\mathbf{x} = (x, y)$, we can express the classification of the point as

$$\text{sign}(ax + by + c)$$

  where a, b, and c are constants that define a line. We'll have to choose these somehow.
- This function will return a $+1$ if $ax + by + c$ is positive and -1 otherwise.

How this Translates Graphically

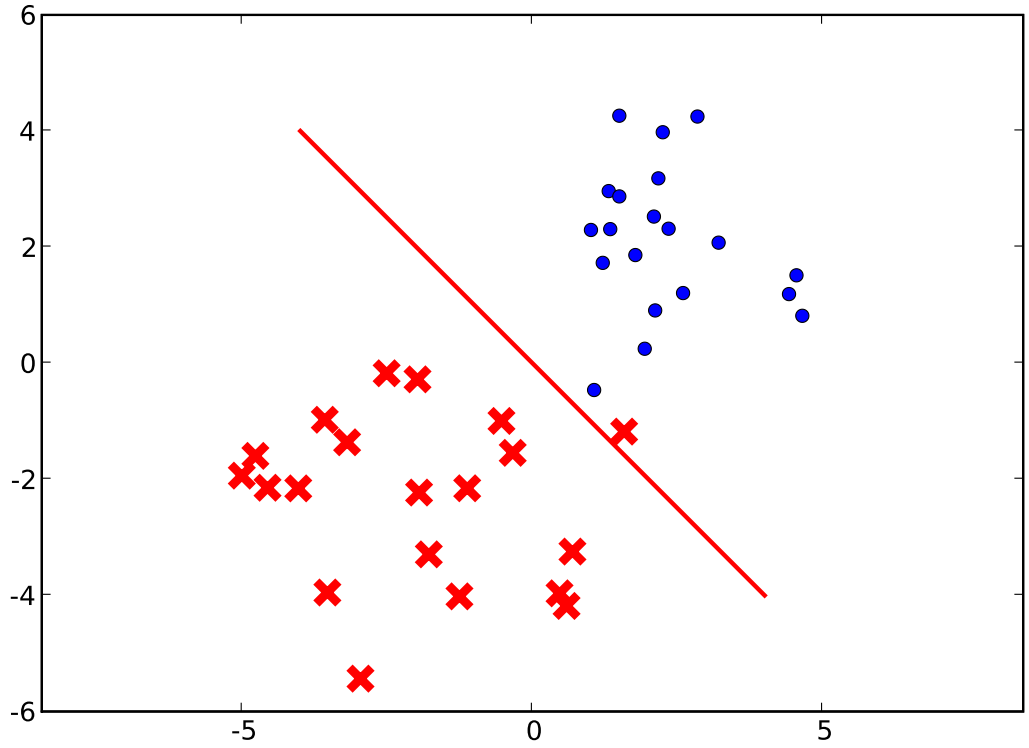
- Effectively, we are projecting every point onto a line.
- Every point projects to some point on the line. The sign of the location along the lines determines the classification of the point



How can we find the separating line?

- This separating line can be found by looking at the line

$$ax + by + c = 0$$

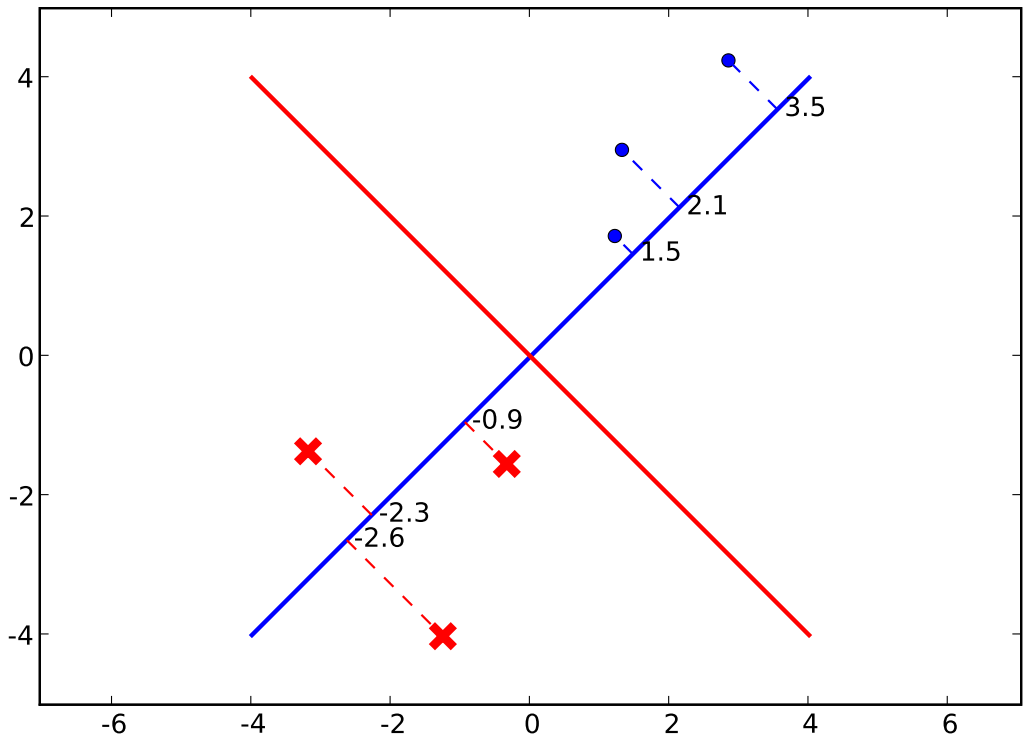


How do we get the parameters of this line?

- We'll find the parameters using a machine learning approach
- We'll form a *training set* of examples, along with the correct classification.
- We'll find the line that best separates the training examples.
- Now, we have to form a set of mathematical steps for finding the line that "best separates" the training set.

Optimizing the parameters of the line

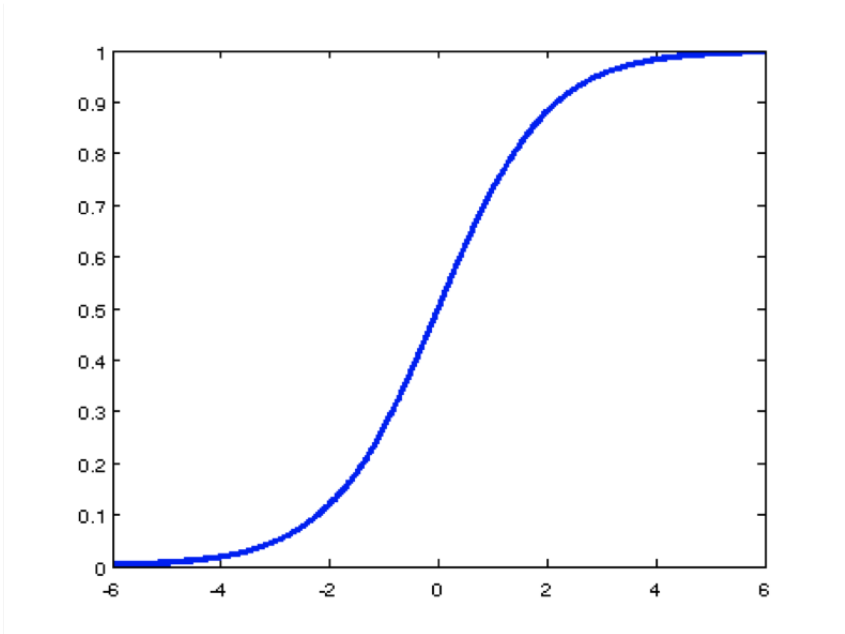
- Notice that the points that are the farthest from the red separating line have the largest response.
- In some sense, we can be more confident in a point's classification as the point gets farther from the separating line.
- So, the bigger the magnitude of a response, the more confident in the classification we can be.



Looking at the confidence in classification

- We can translate this response into a probability.
- We will use the *logistic function* to transform the response into a probability of the correct classification
- We will assume that each point will be labeled with a label l. l can take values $+1$ or -1.

$$P[l = +1|\mathbf{x}] = \frac{1}{1 + \exp(-(ax + by + c))}$$



Finding the line parameters

- Now, for any set of line constants, we can find out the probability assigned to the correct label of each item in the training set.

$$\prod_{i=1}^{N} P[l_i|\mathbf{x}_i] = \prod_{i=1}^{N} \frac{1}{1+\exp(-l_i(ax_i+by_i+c))}$$

- We've inserted l_i into the exponent because, if $l_i = -1$

$$P[l=-1|\mathbf{x}] = \frac{1}{1+\exp((ax+by+c))}$$

- We multiply the probabilities because we believe the points are drawn independently.
- Note that for each item, this number tells us how much the model defined by that particular line believes that the ground-truth right answer is the actual right answer.

Finding the line

$$\prod_{i=1}^{N} P[l_i|\mathbf{x}_i] = \prod_{i=1}^{N} \frac{1}{1+\exp(-l_i(ax_i+by_i+c))}$$

- This is also called the likelihood of the data
- Ideally, this likelihood should be as close to 1 as possible.
- We can find the parameters of the line by looking for the line parameters that maximize this likelihood.
- In other words, find the set of line parameters that make the right answers have as high a probability as possible.

Finding the line

$$\prod_{i=1}^{N} P[l_i|\mathbf{x}_i] = \prod_{i=1}^{N} \frac{1}{1+\exp(-l_i(ax_i+by_i+c))}$$

- Before going on, we are going to do a quick math trick.
- Because the $\log$ (natural logarithm or $\ln$) function is monotonically increasing (i.e. it is always increasing), the parameters that maximize the above equation will also maximize

$$\log\left(\prod_{i=1}^{N} P[l_i|\mathbf{x}_i]\right) = \sum_{i=1}^{N} -\log\left(1+\exp(-l_i(ax_i+by_i+c))\right)$$

Finding the line

$$\log\left(\prod_{i=1}^{N} P[l_i|\mathbf{x}_i]\right) = \sum_{i=1}^{N} -log\left(1+\exp(-l_i(ax_i+by_i+c))\right)$$

- Similarly, we can multiply this equation by -1 to change from a *maximization* problem to a *minimization* problem
- Leading to a function that we will call a "loss function" or "cost function" and can be denoted as L:

$$L = \sum_{i=1}^{N} \log\left(1+\exp(-l_i(ax_i+by_i+c))\right)$$

- Our goal is to find the line parameters that minimize L.

Minimizing L

- We can minimize L by using its gradient:

$$\nabla L = \begin{bmatrix} \frac{\partial L}{\partial a} \\ \frac{\partial L}{\partial b} \\ \frac{\partial L}{\partial c} \end{bmatrix}$$

- The gradient can be viewed as a vector that points in the direction of steepest ascent.
- So, the negative gradient points in the direction of steepest descent.

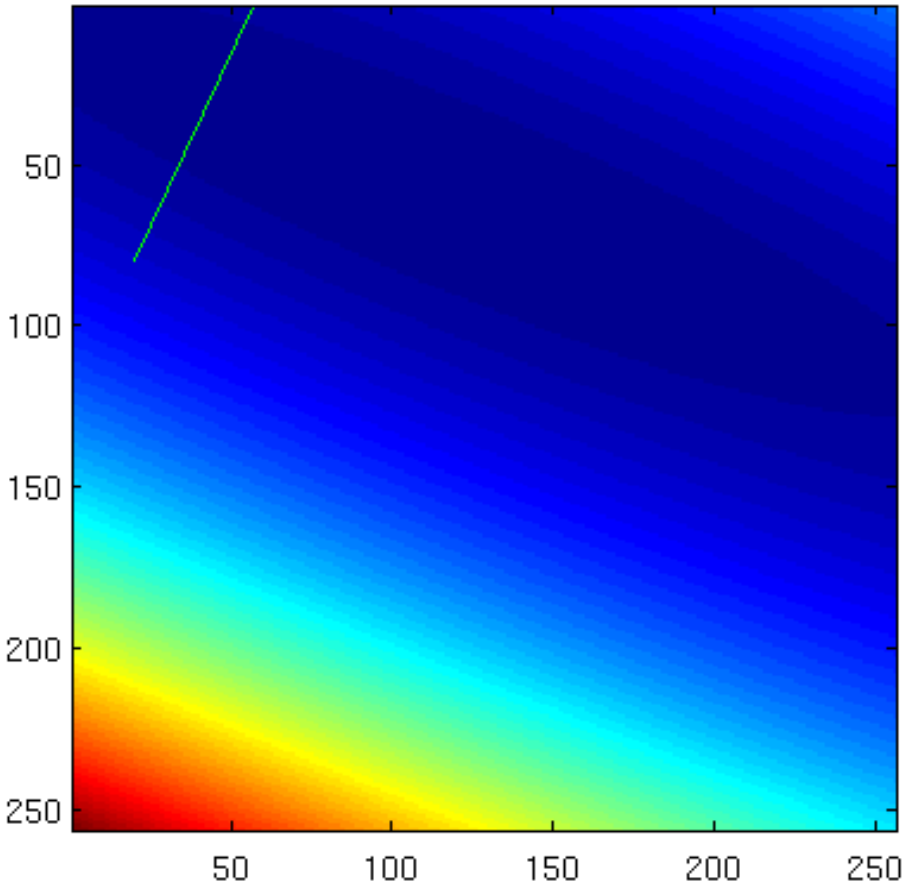
An algorithm for minimizing L

- This leads to a simple algorithm for optimizing the line parameters.
- We'll create a vector

$$\mathbf{p} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$$

- The steps are:
  1. Initialize $\mathbf{p}$ to some value $\mathbf{p}_0$
  2. Repeat these steps:
     - 2.1 Calculate ∇L using the current value of $\mathbf{p}$ (The value of the gradient depends on $\mathbf{p}$!)
     - 2.2 $\mathbf{p} \leftarrow \mathbf{p} + \eta(-\nabla L)$
- We go in the direction of the negative gadient because that this the direction of steepest descent.

An Example



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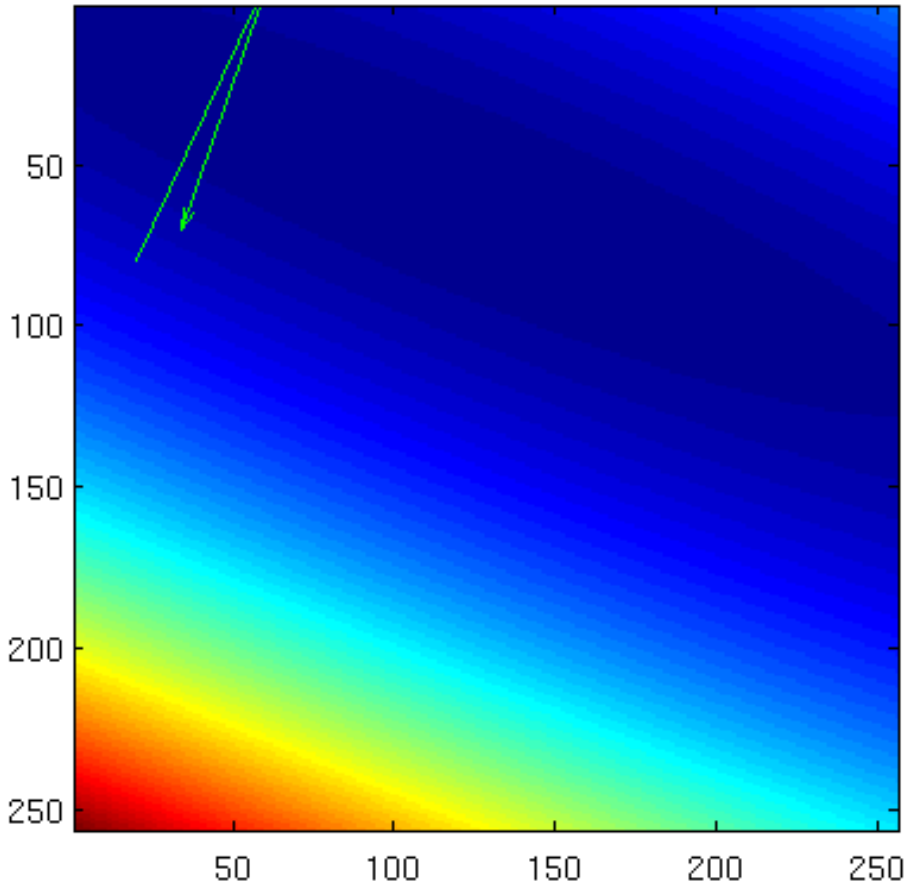

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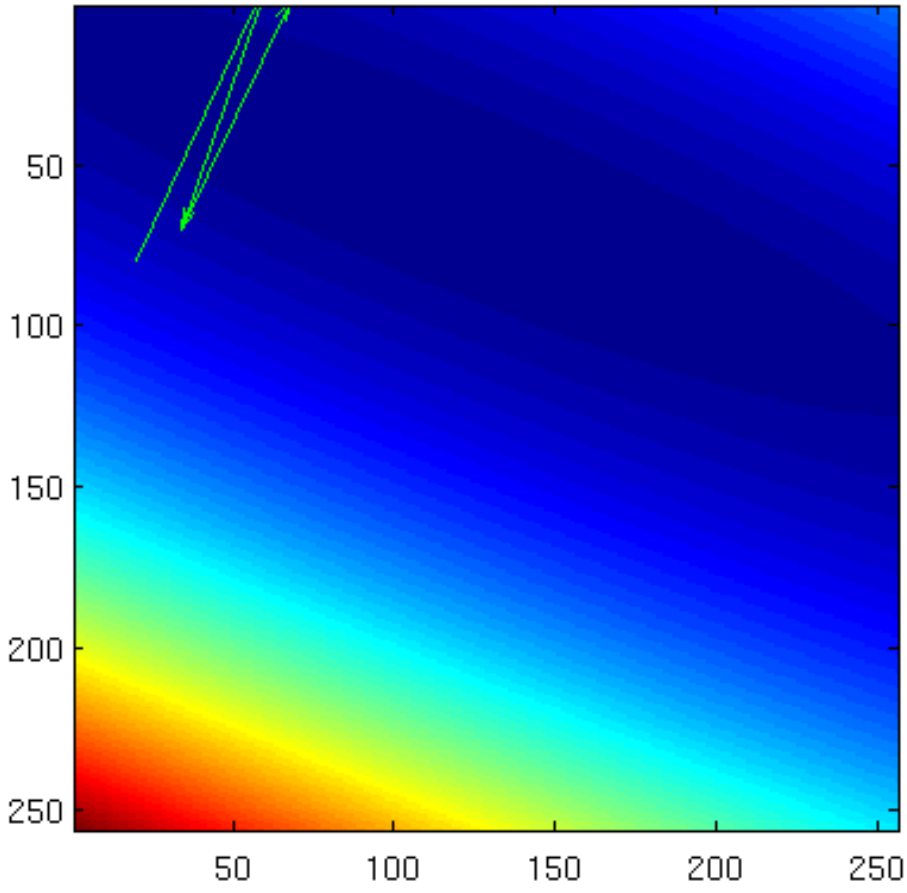

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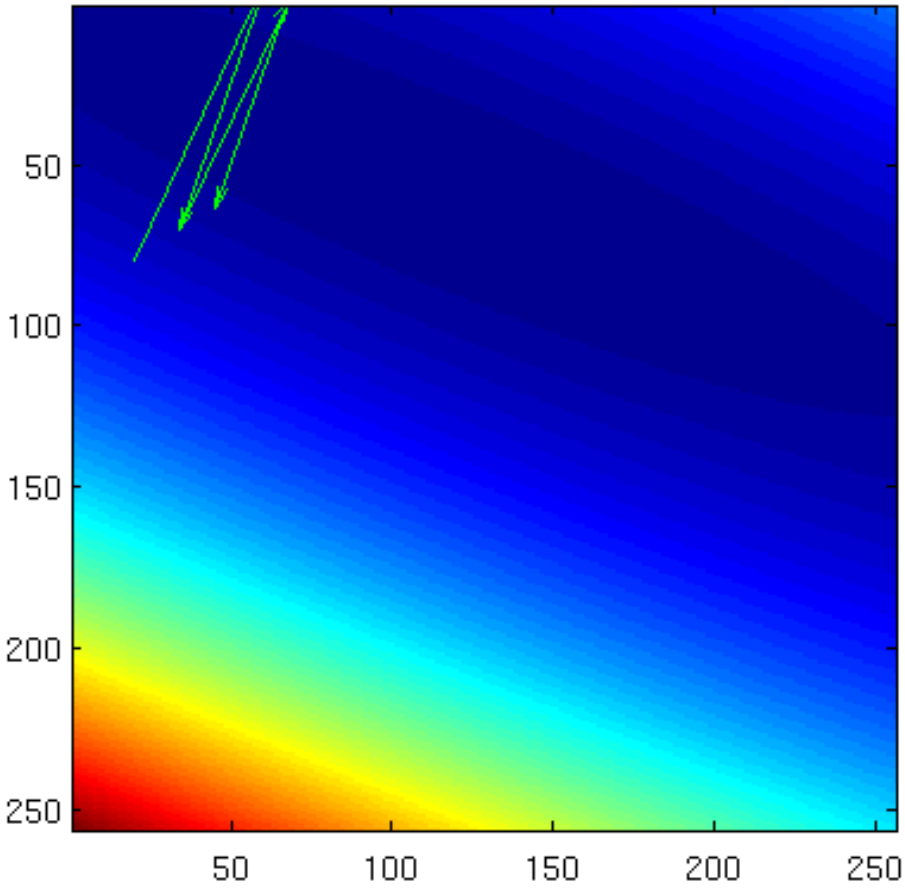

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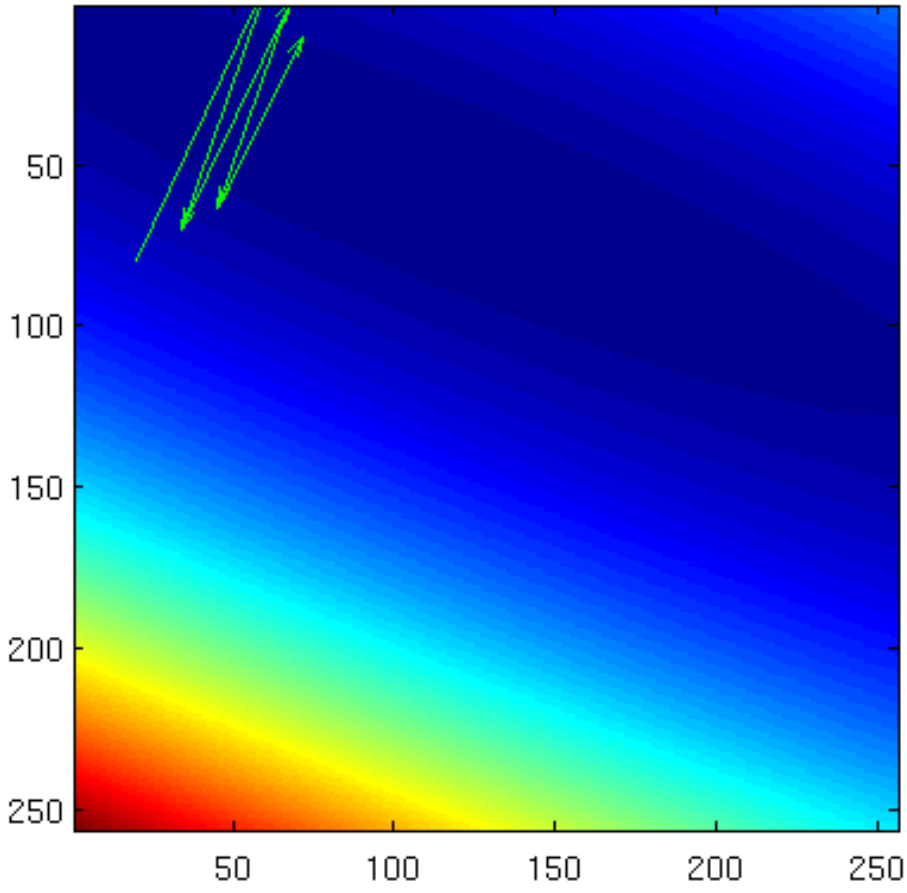

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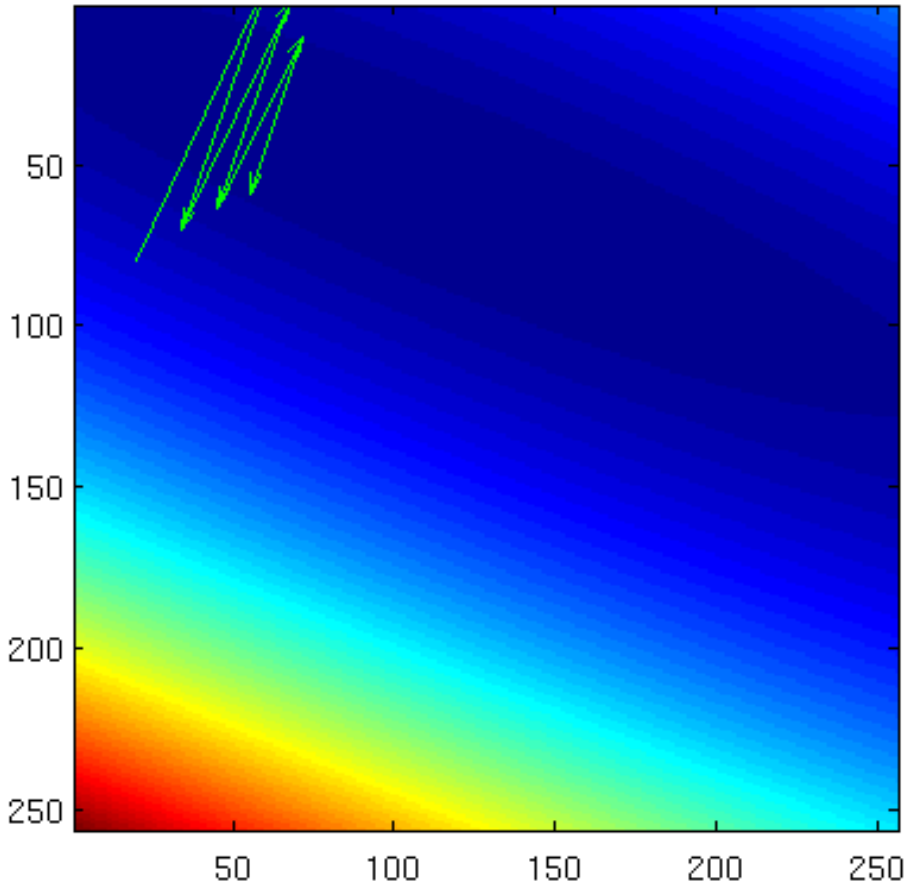

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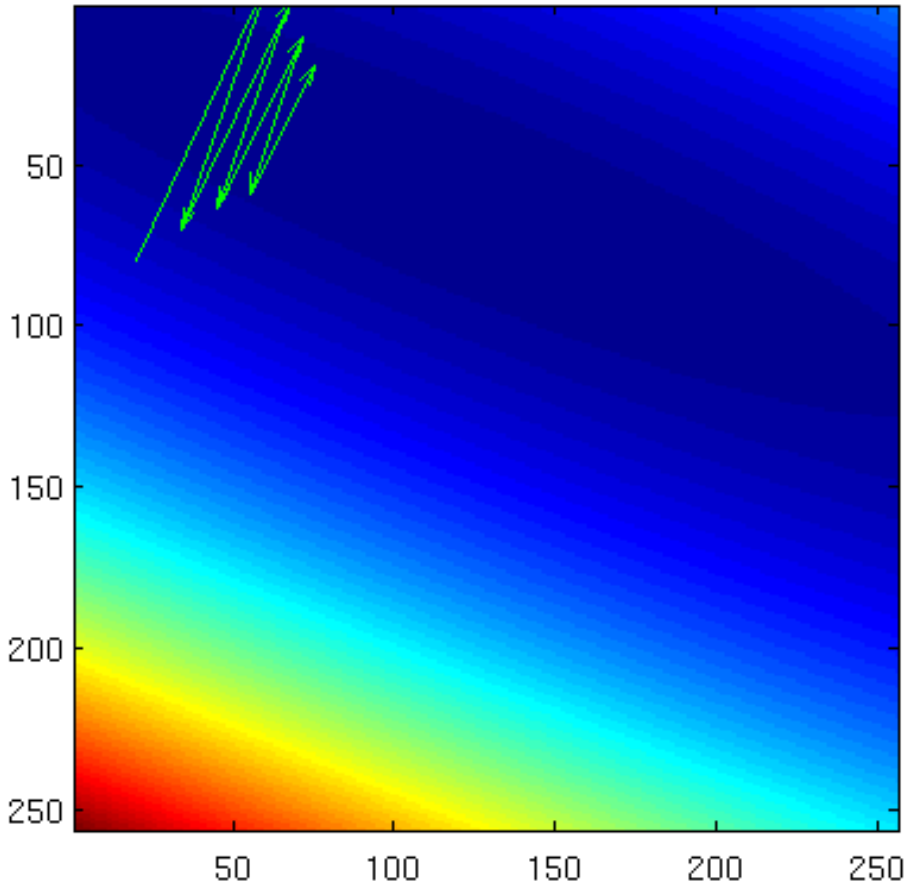

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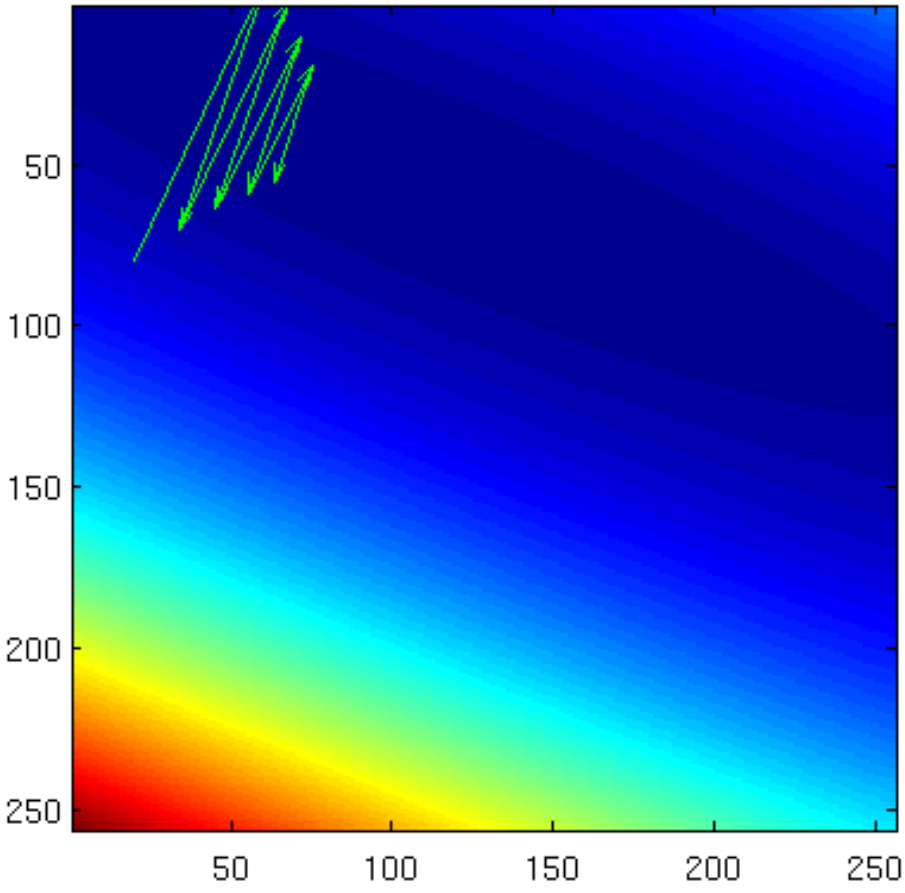

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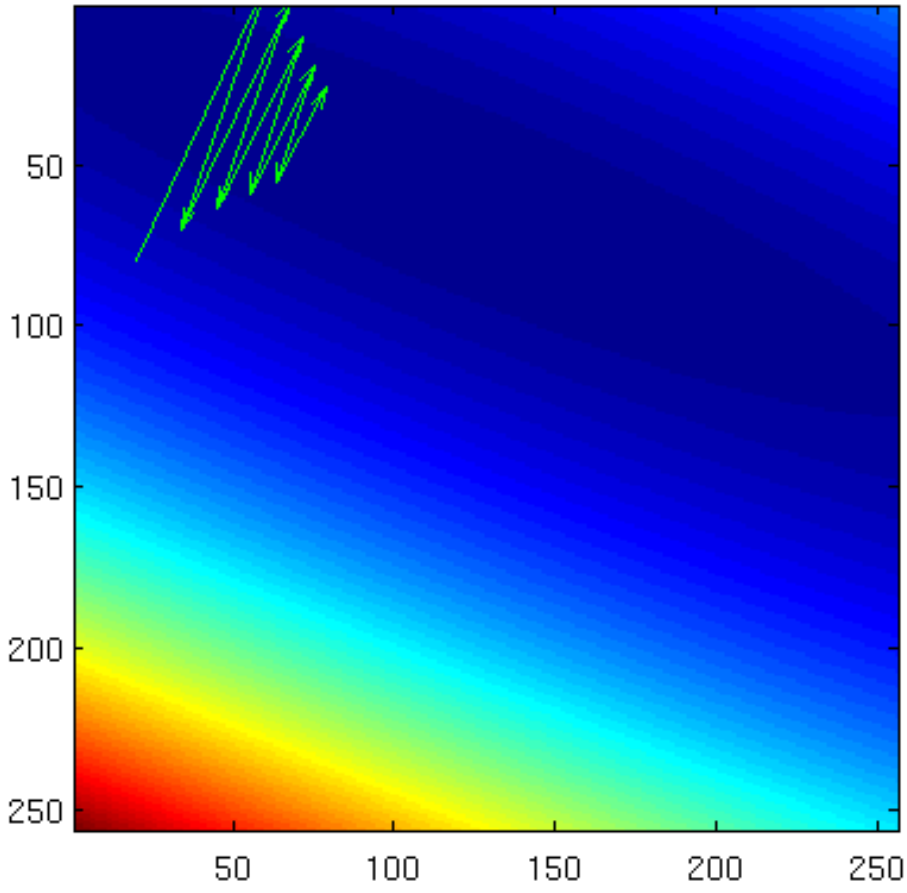

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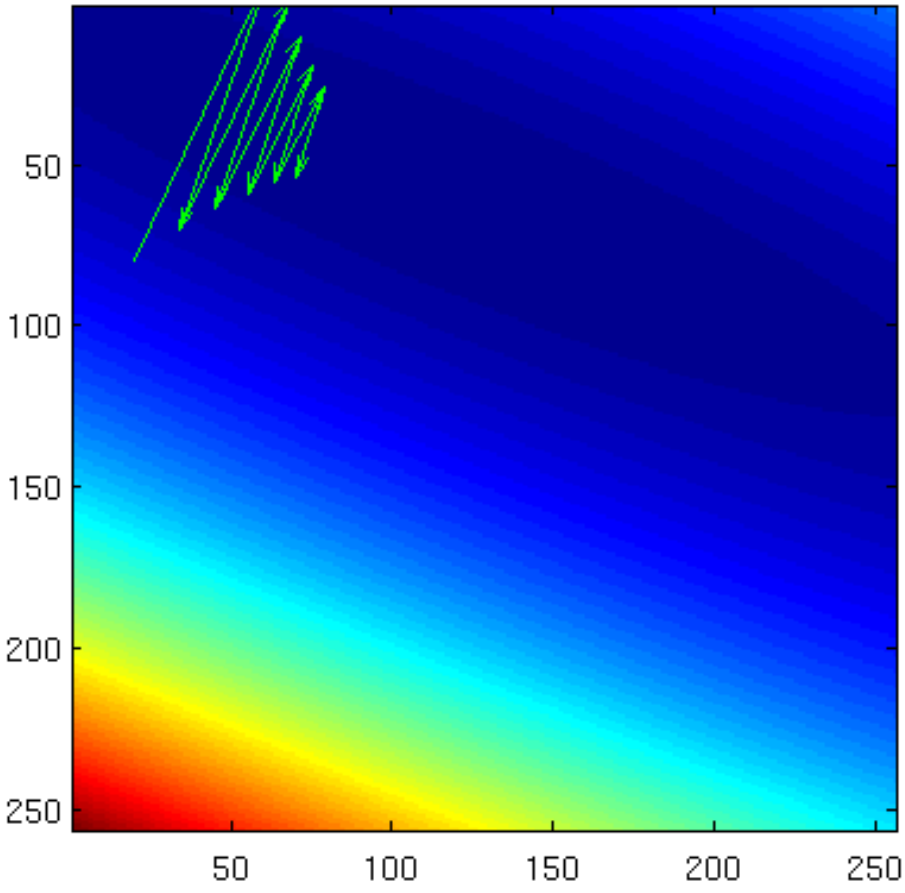

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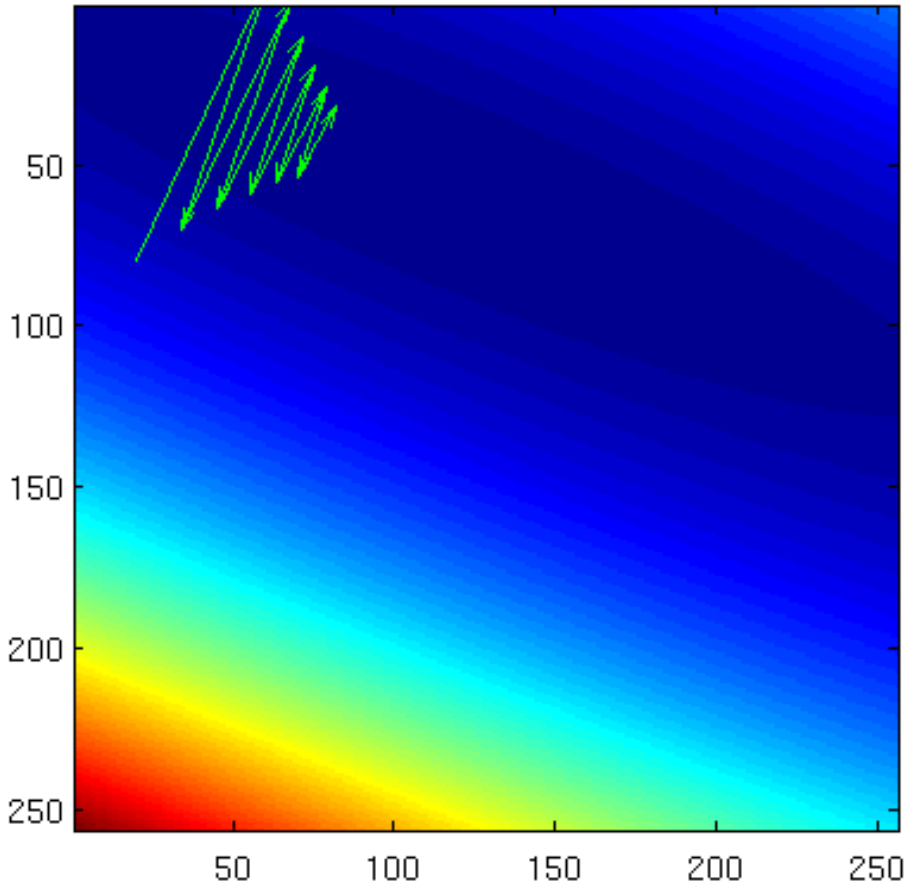

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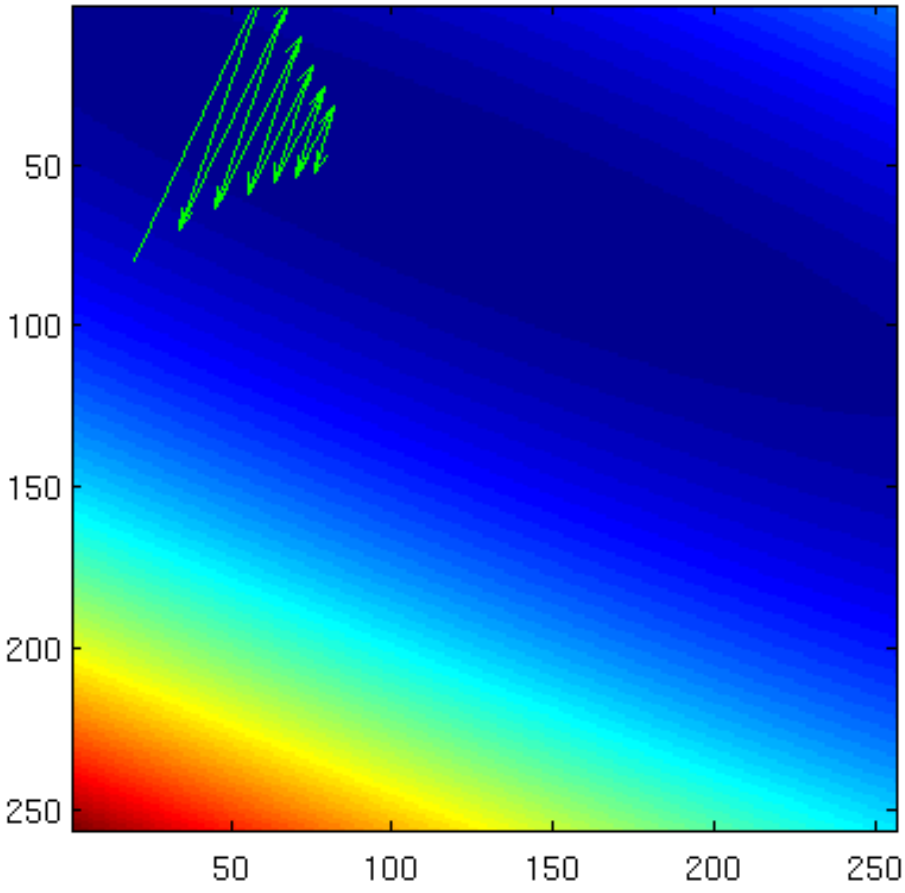

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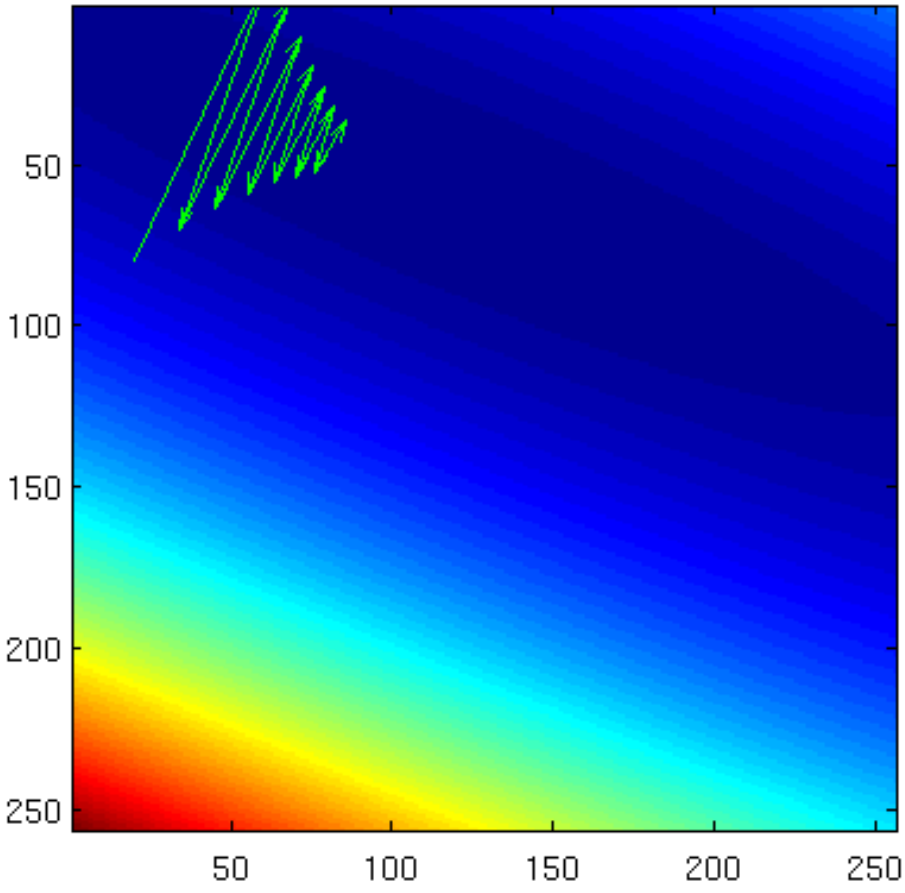

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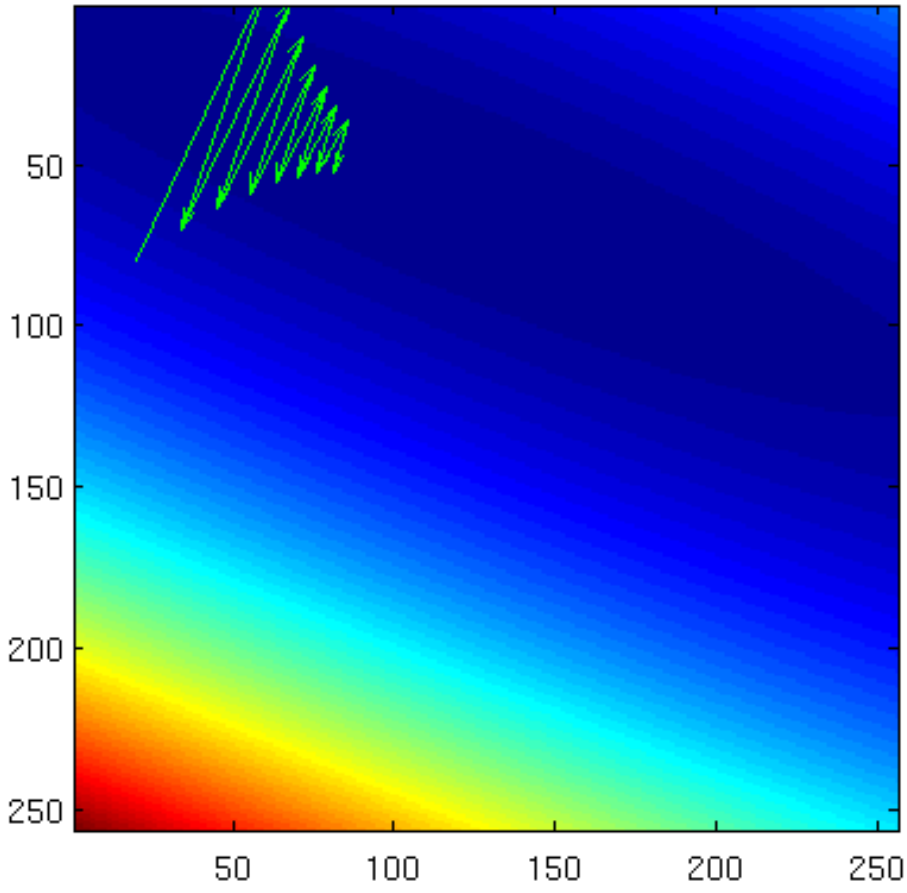

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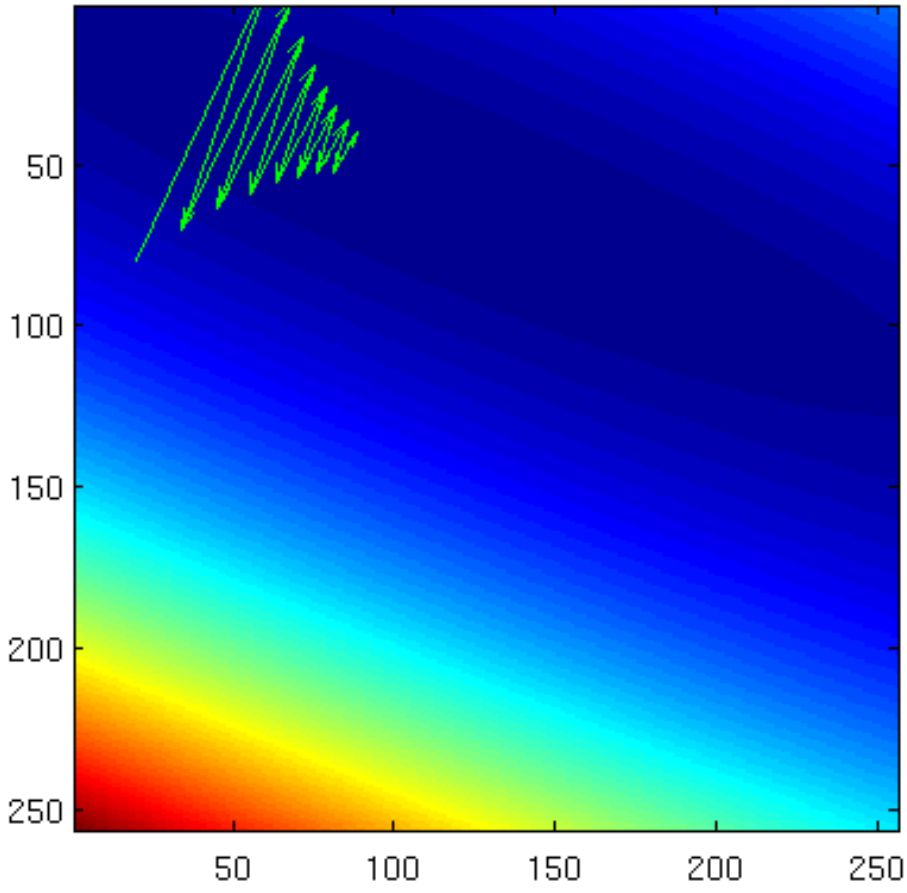

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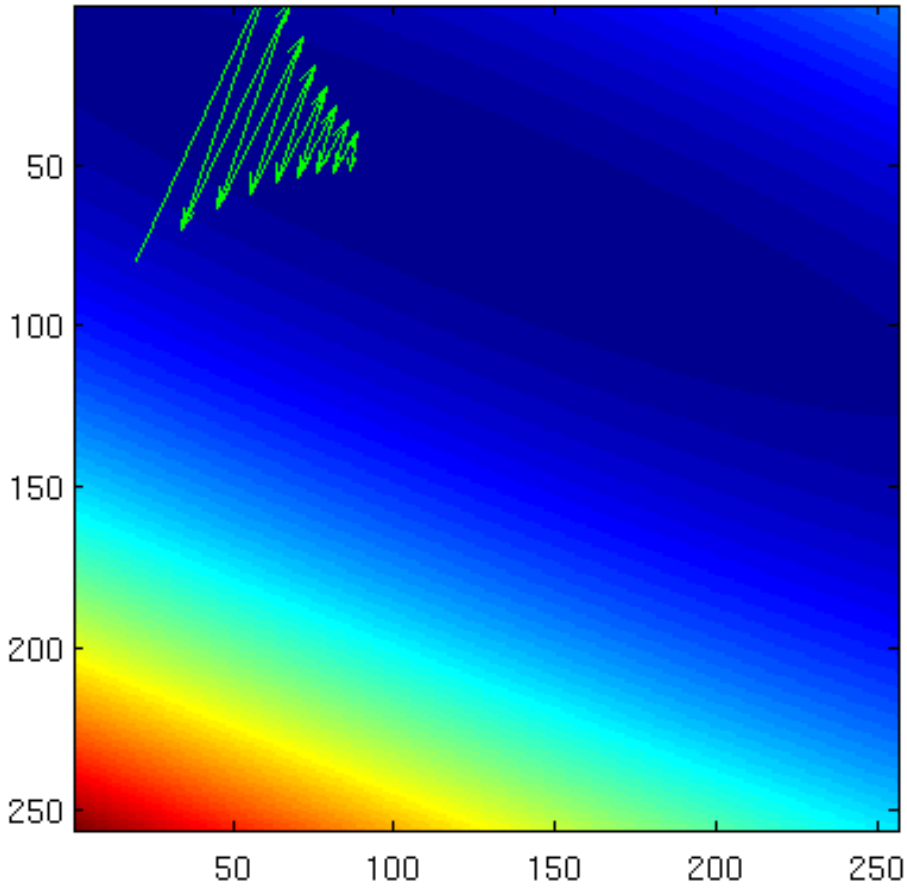

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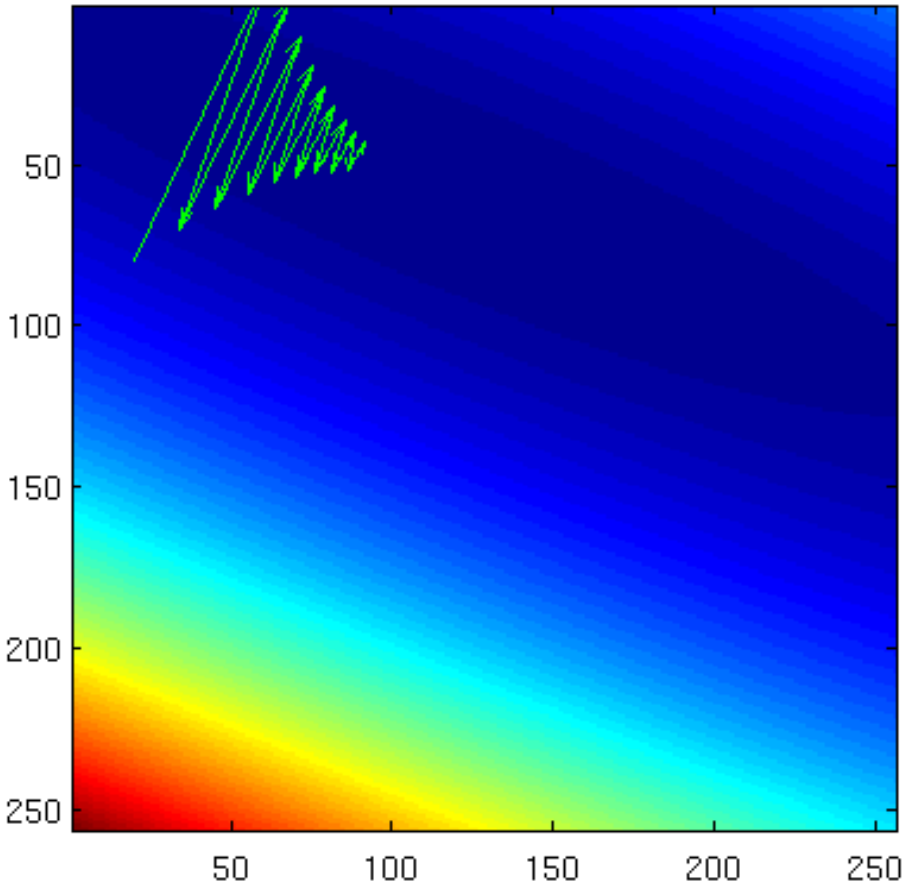

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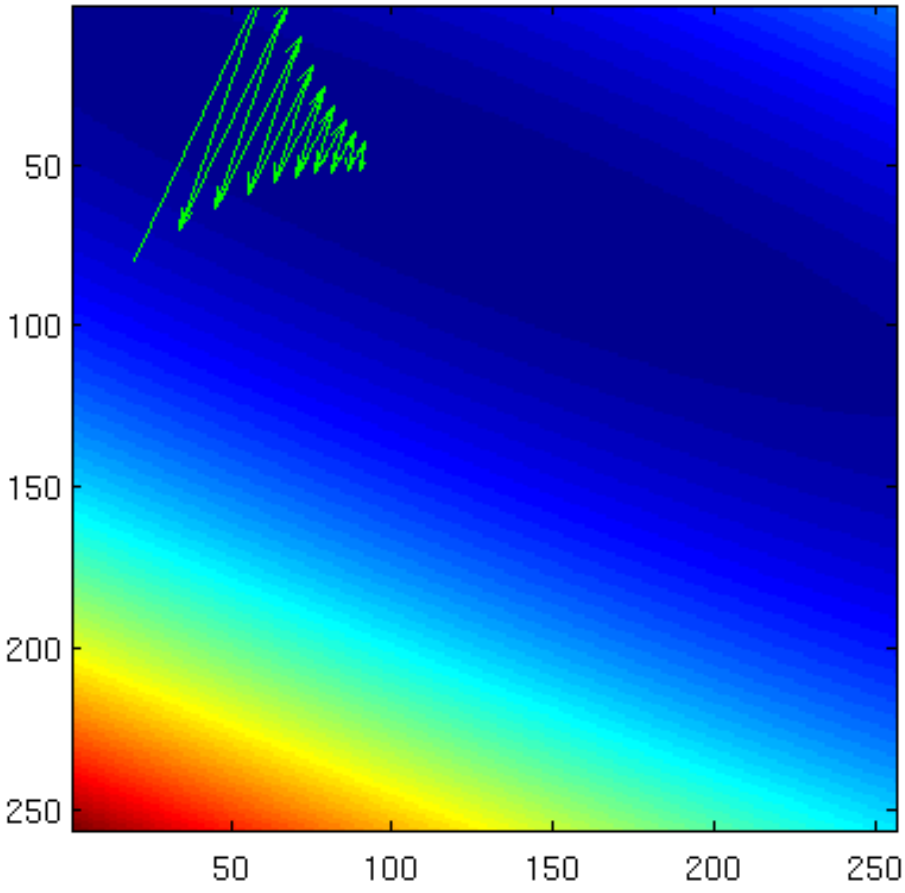

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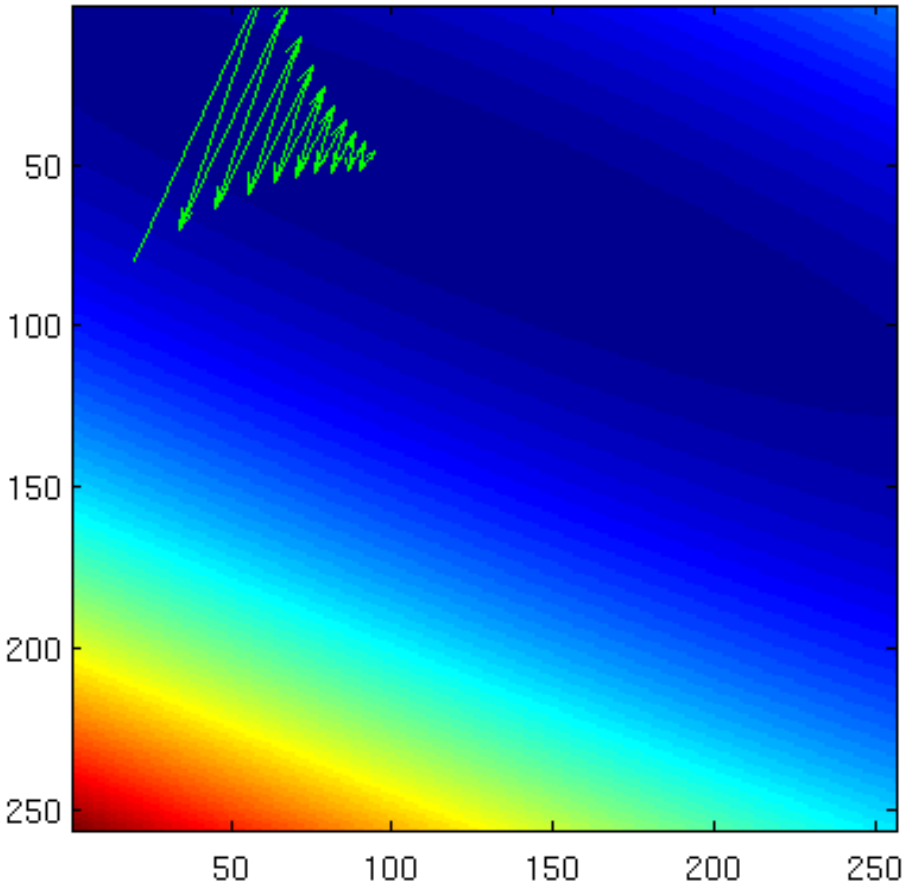

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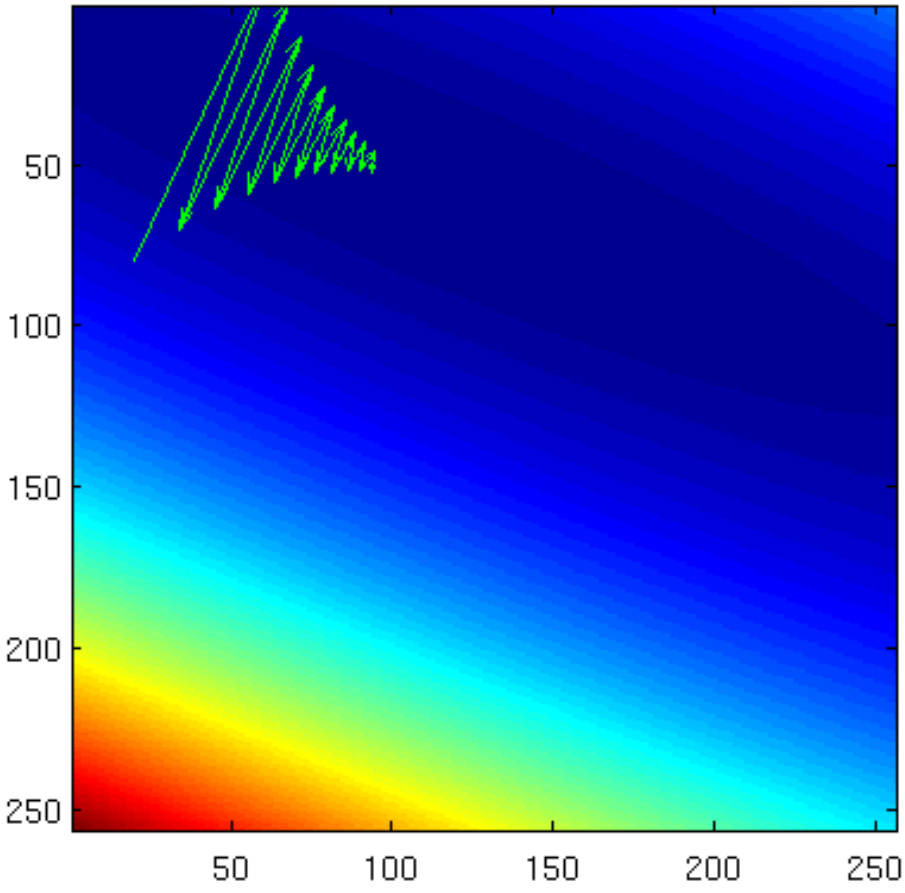

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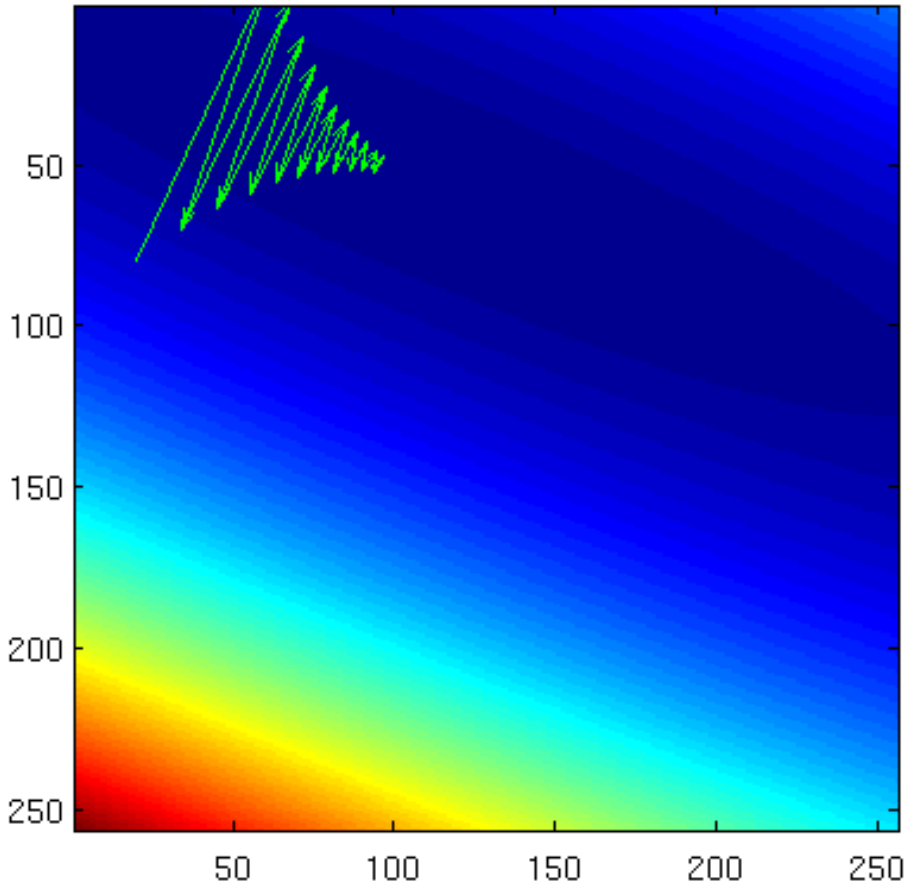

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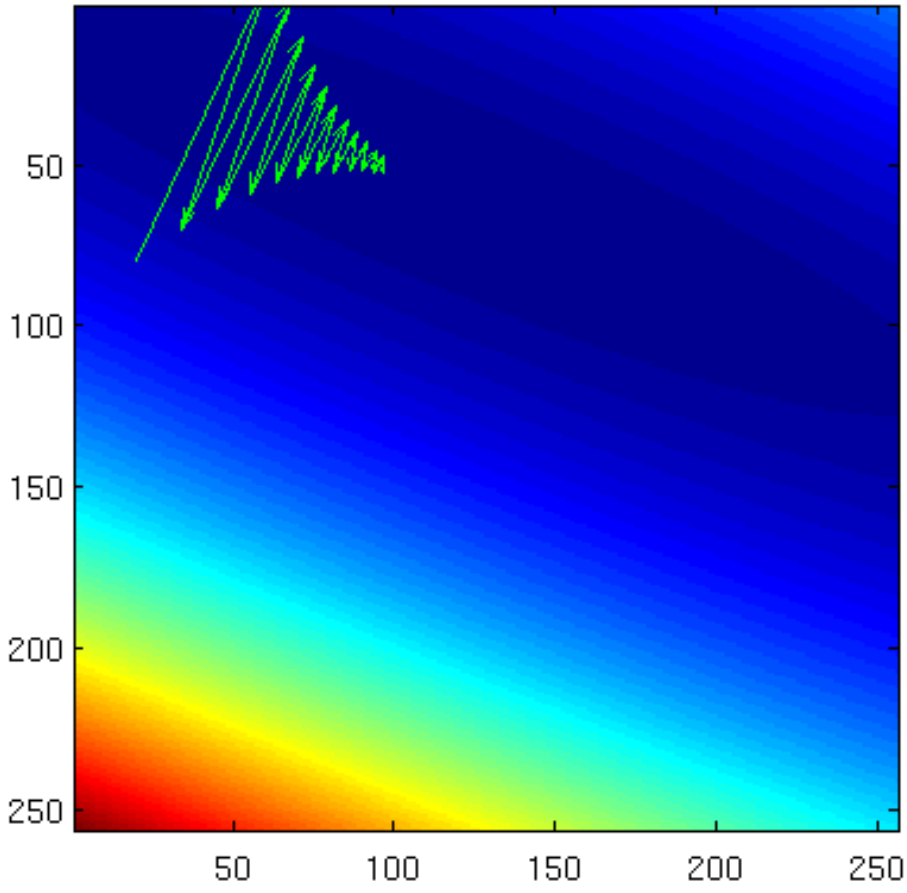

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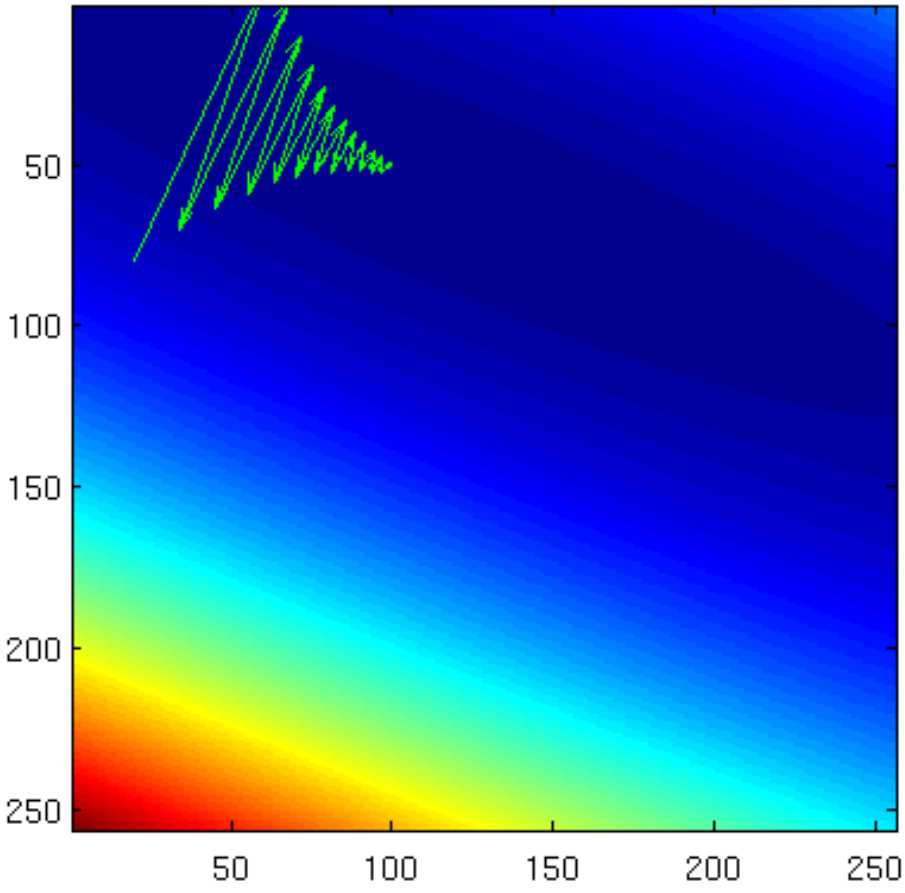

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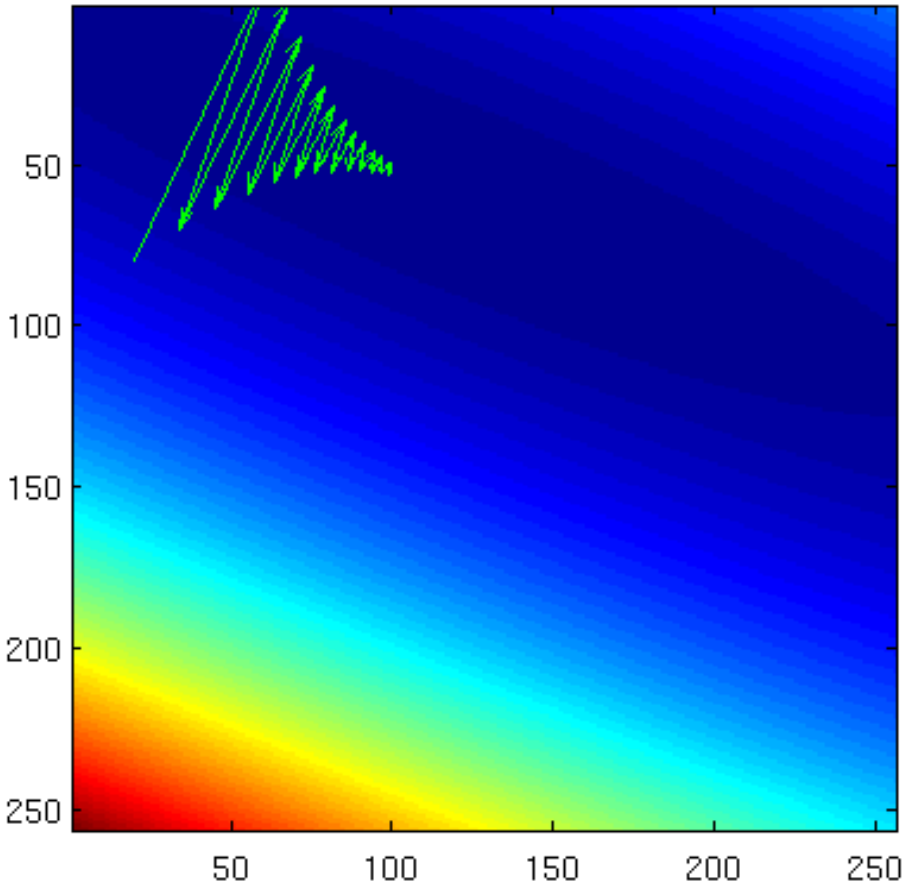

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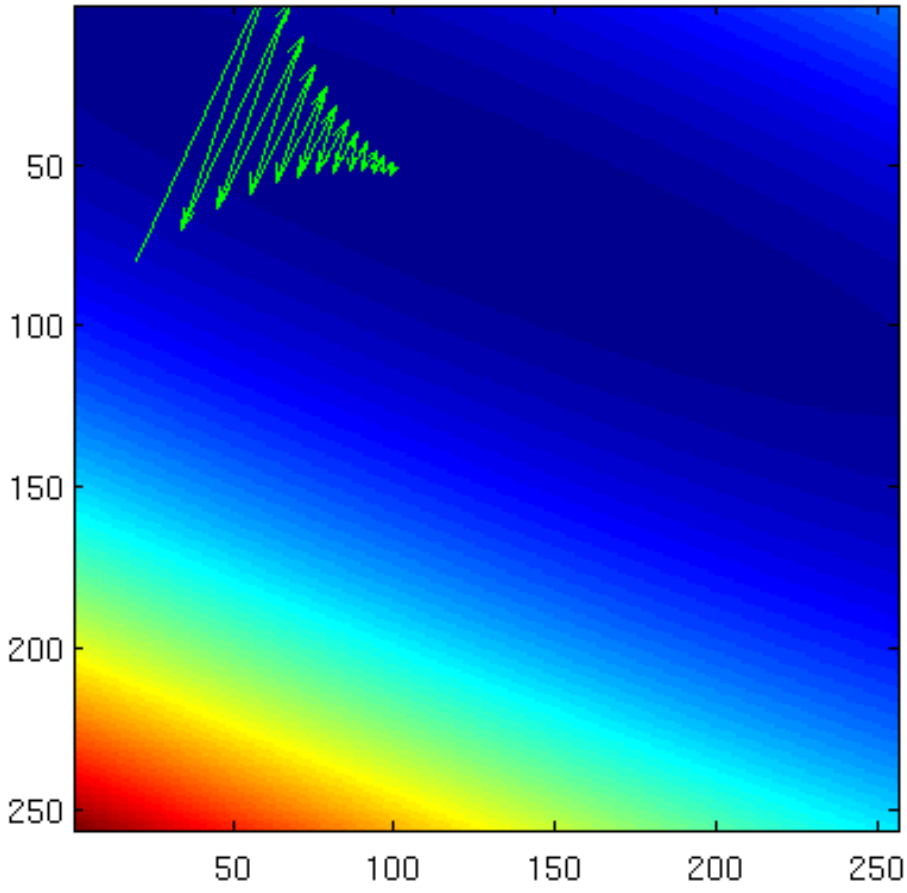

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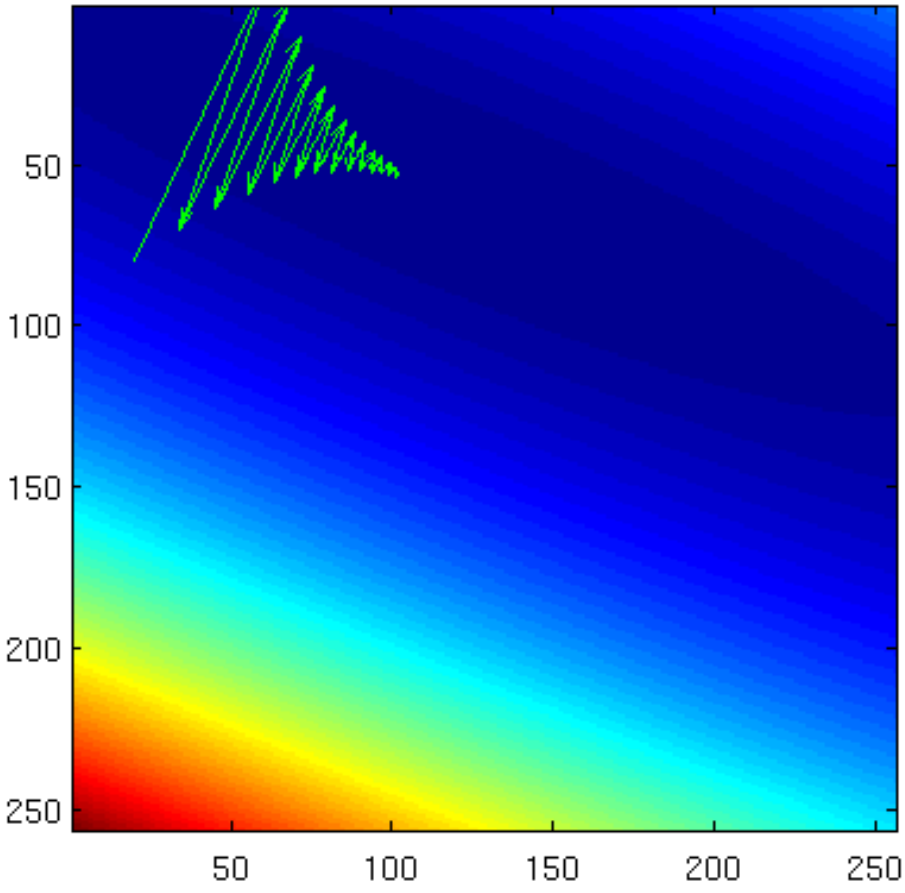

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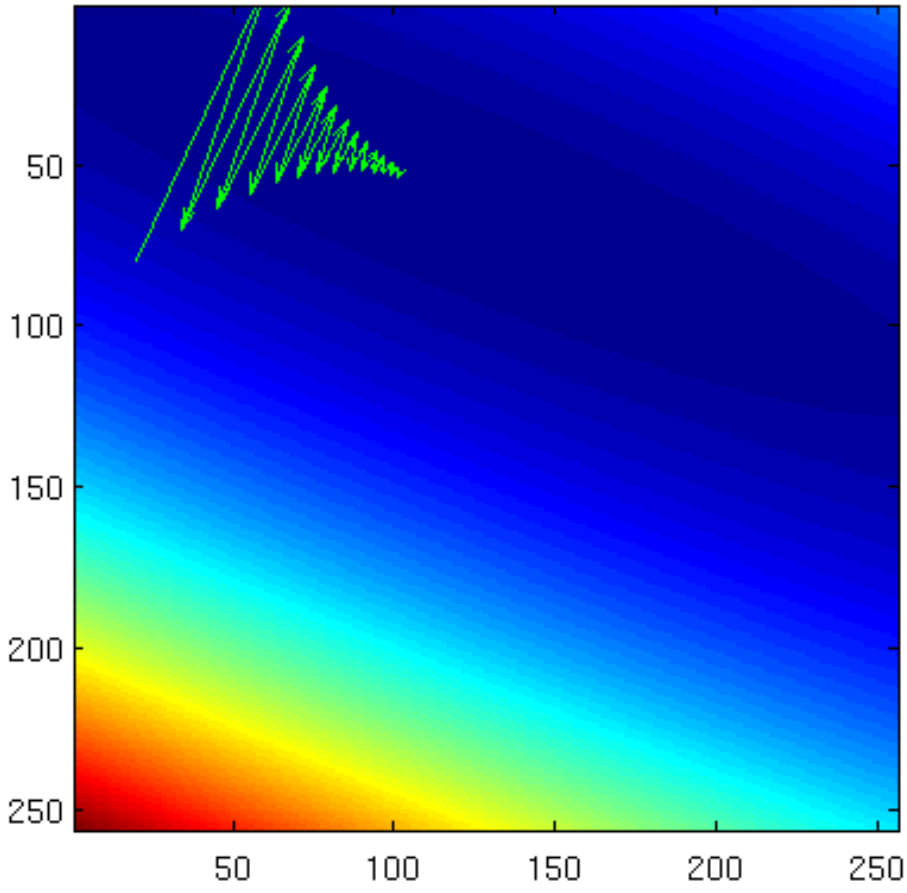

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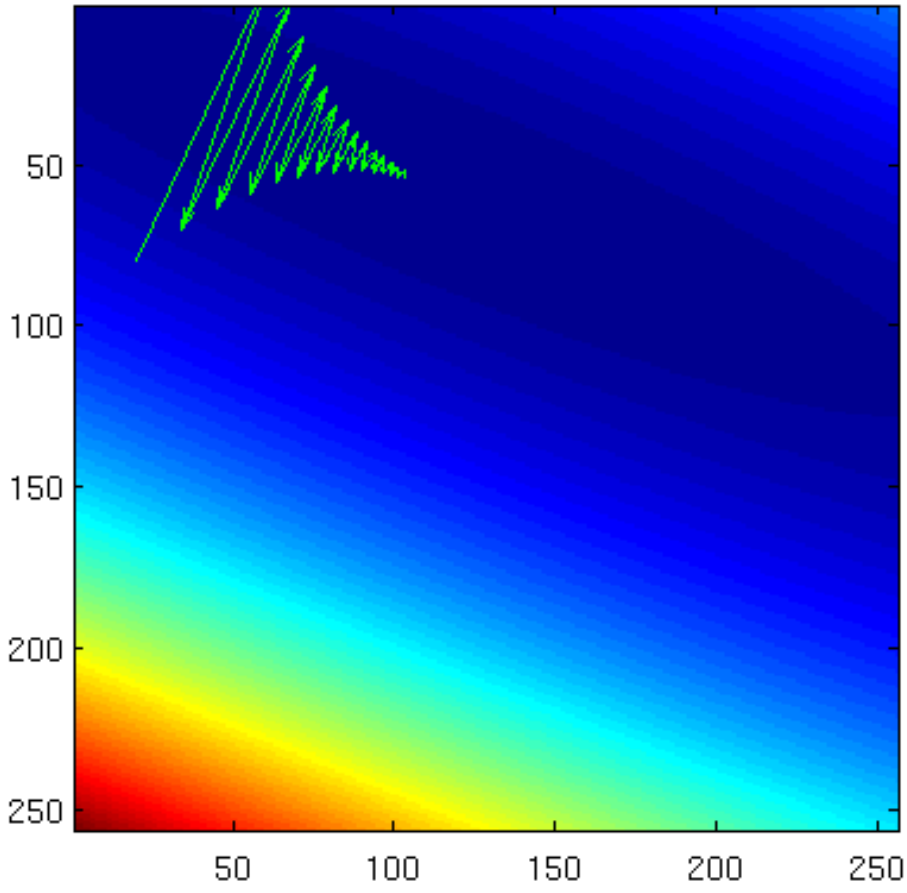

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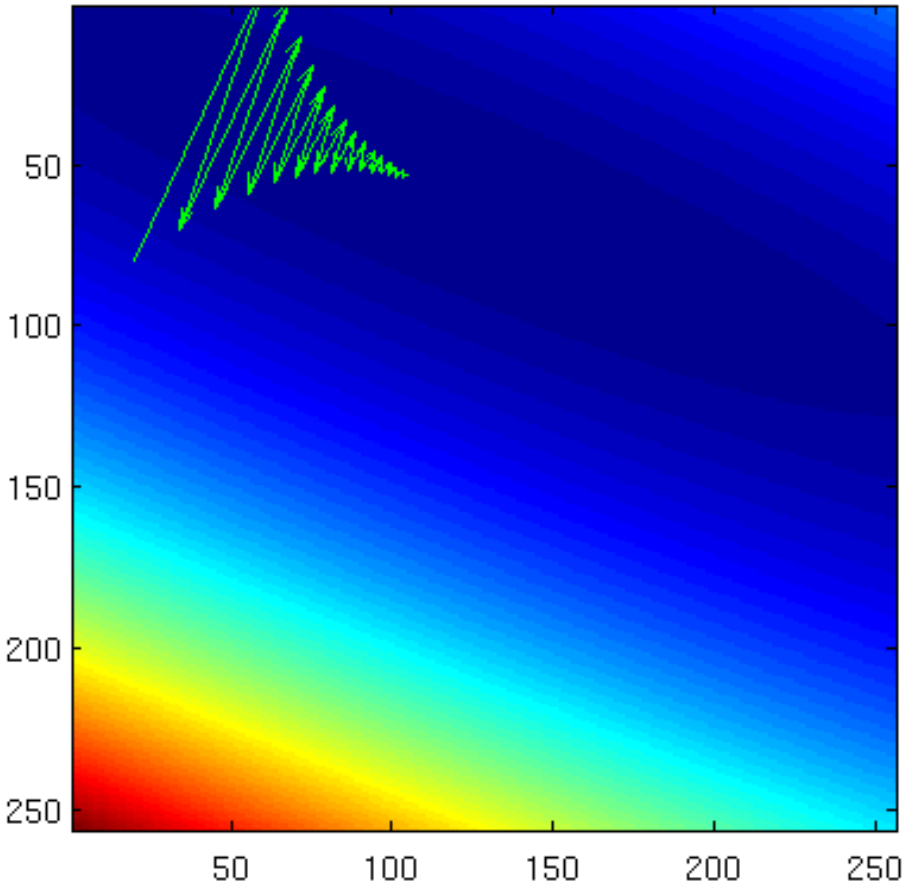

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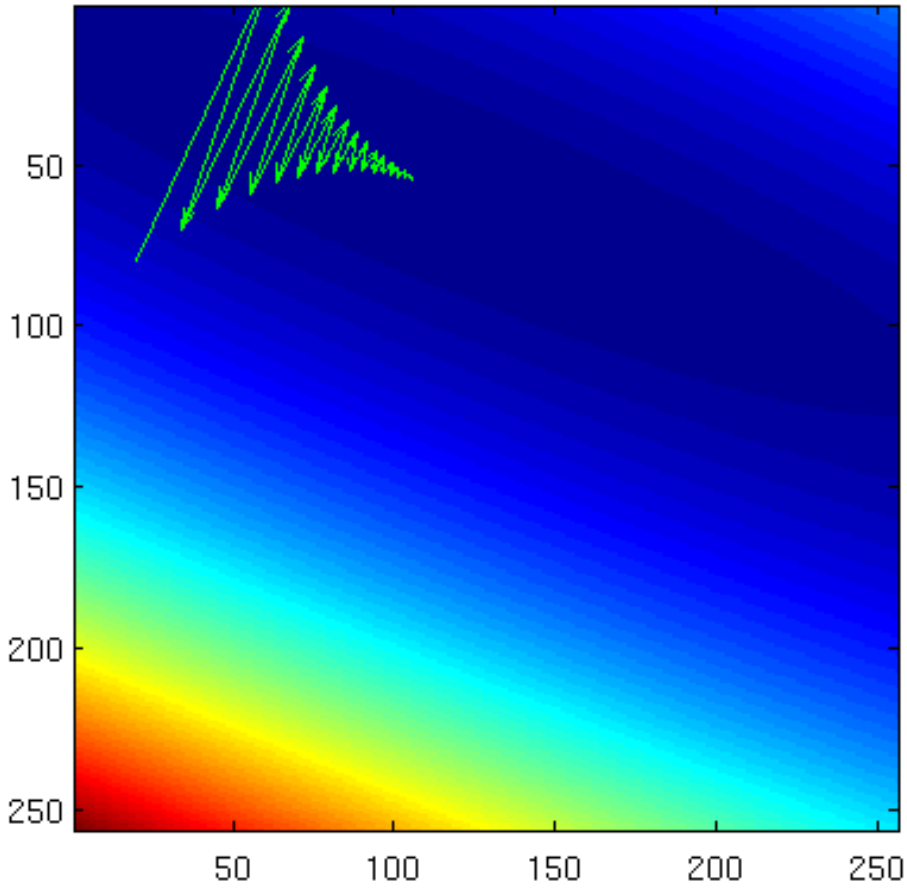

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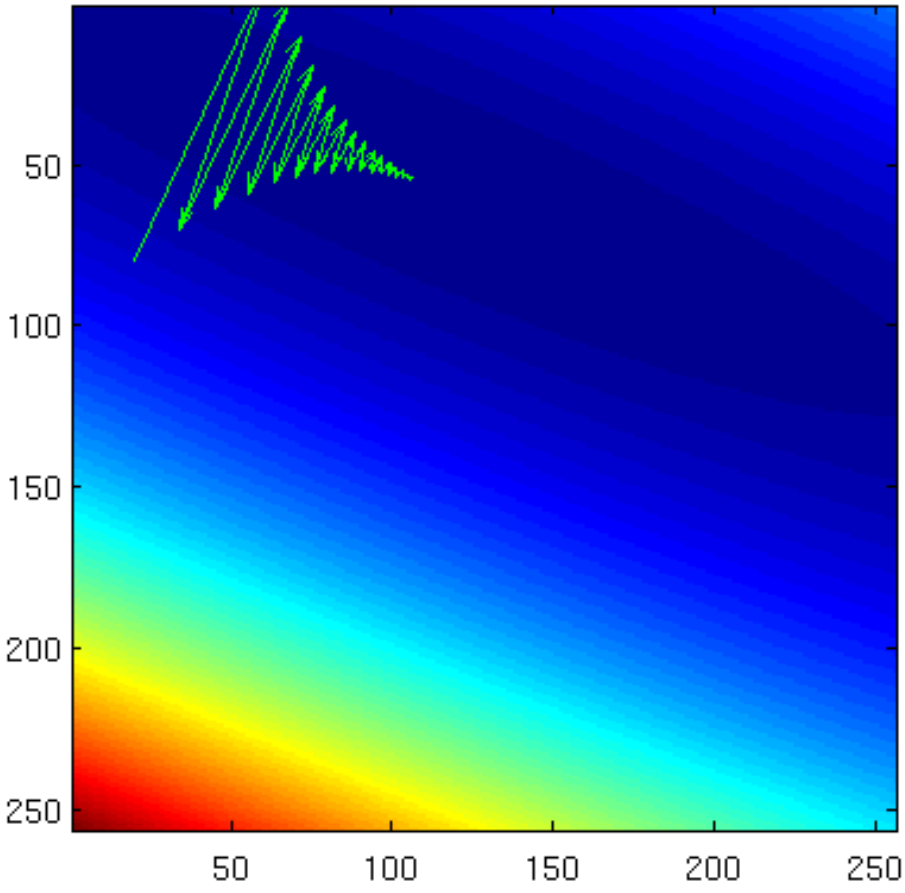

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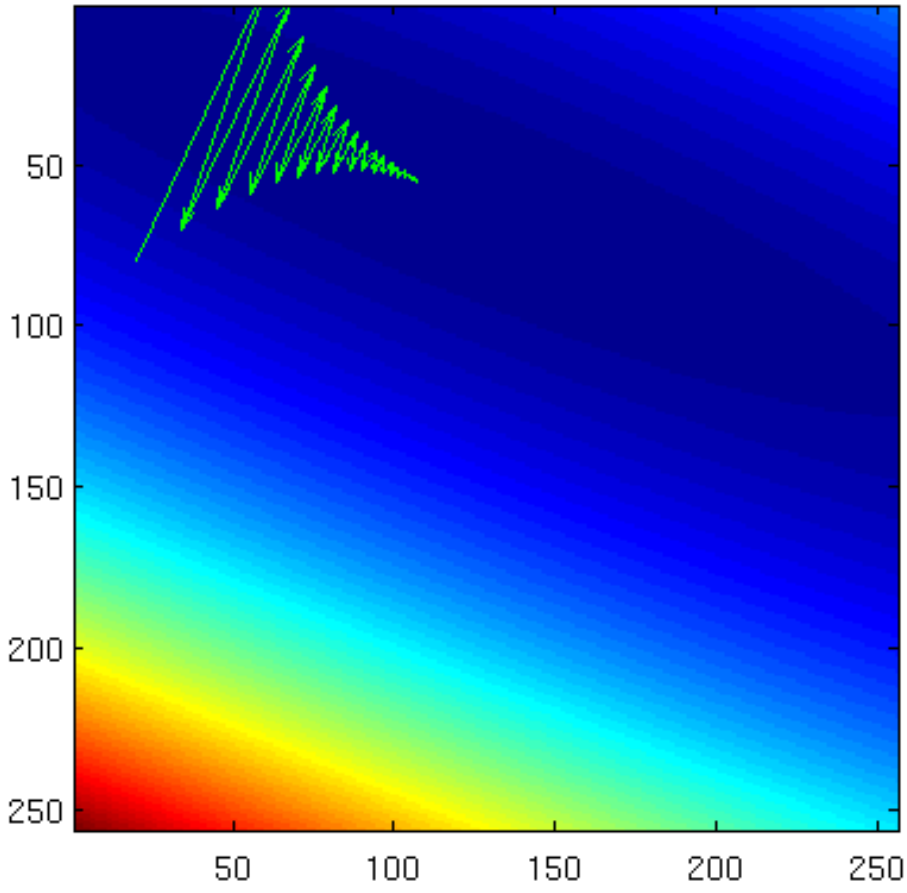

Learning Line parameters

- How does this relate to learning line parameters?
- We differentiate with respect L and optimize.
- The following few slides show how the line changes as the parameters change.

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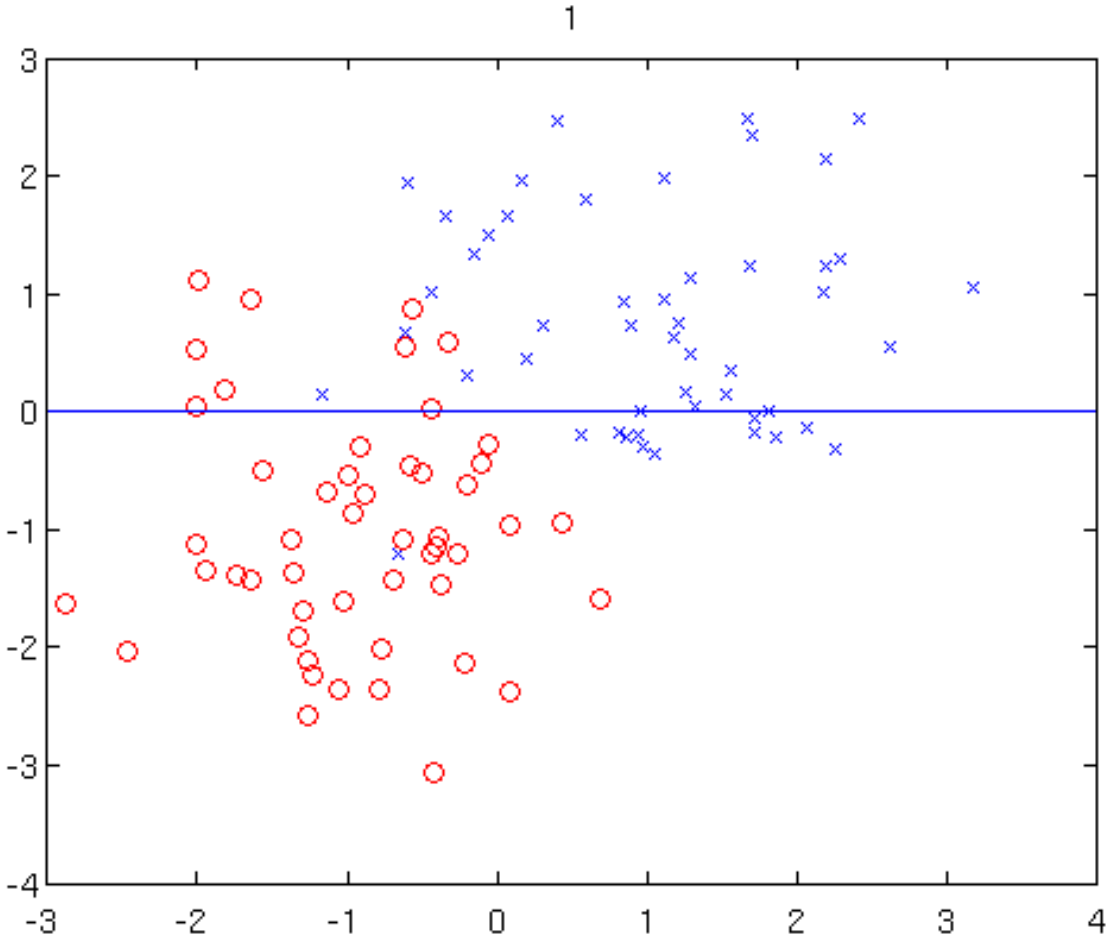

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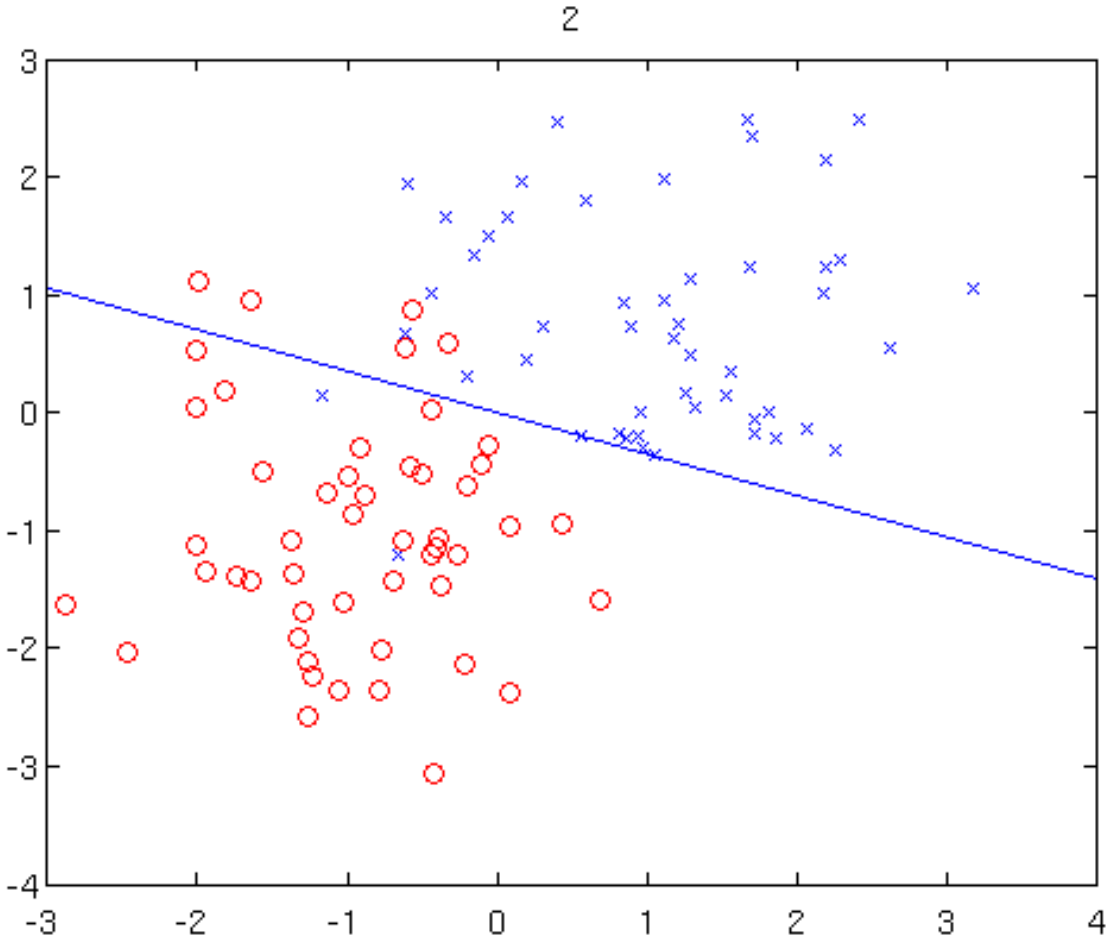

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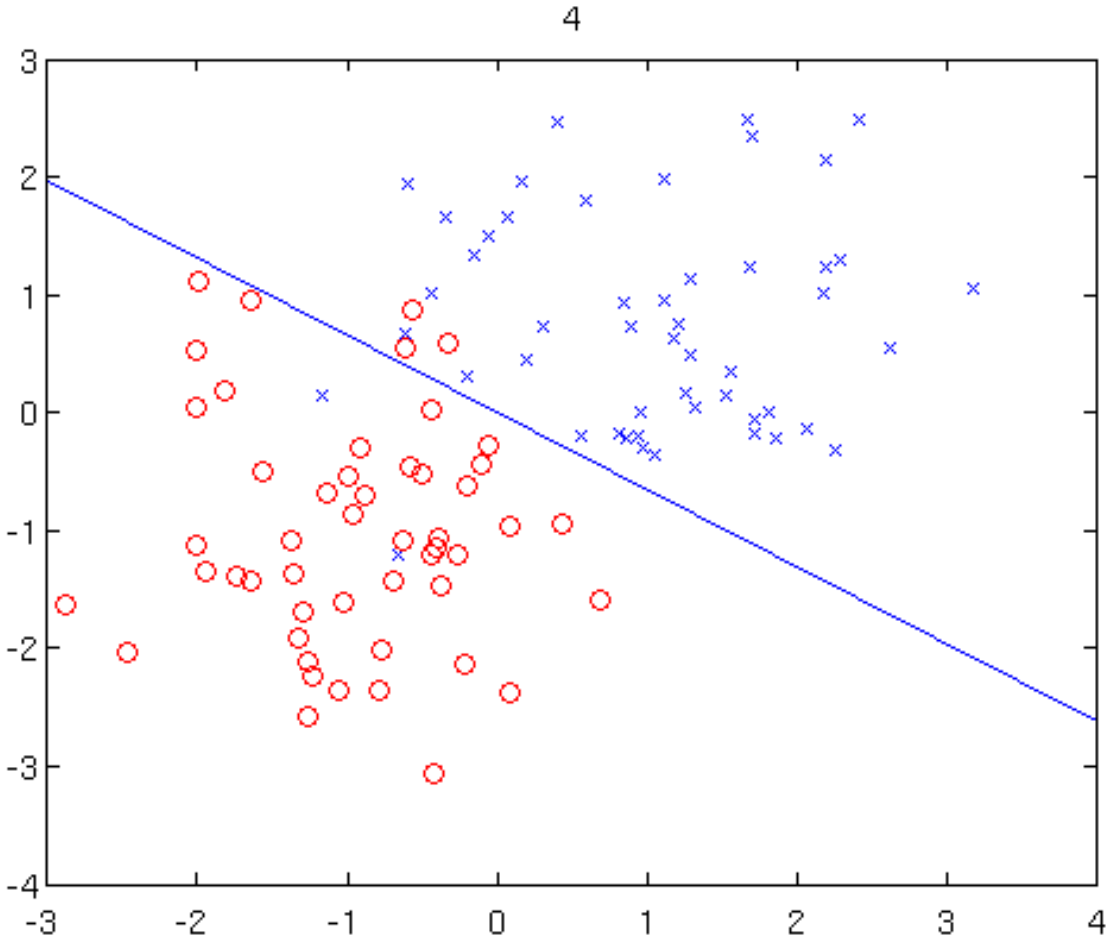

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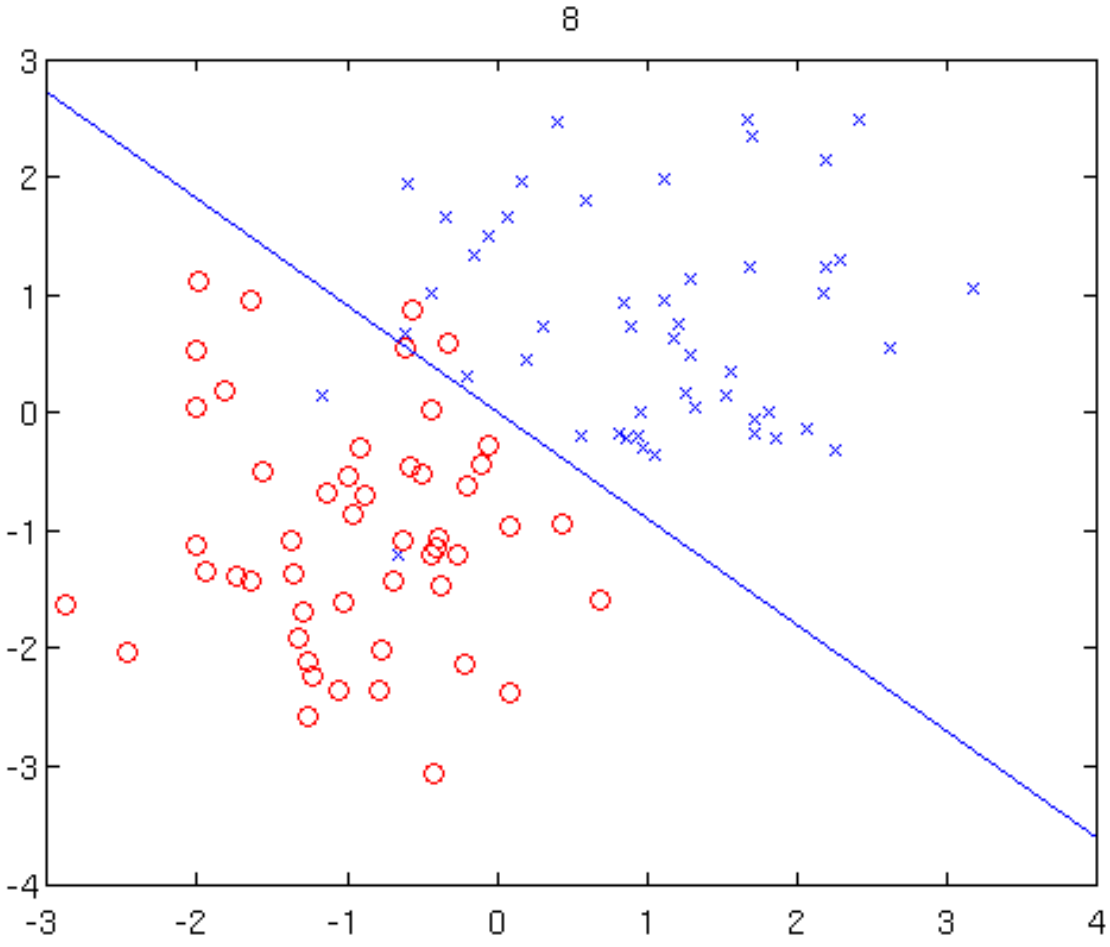

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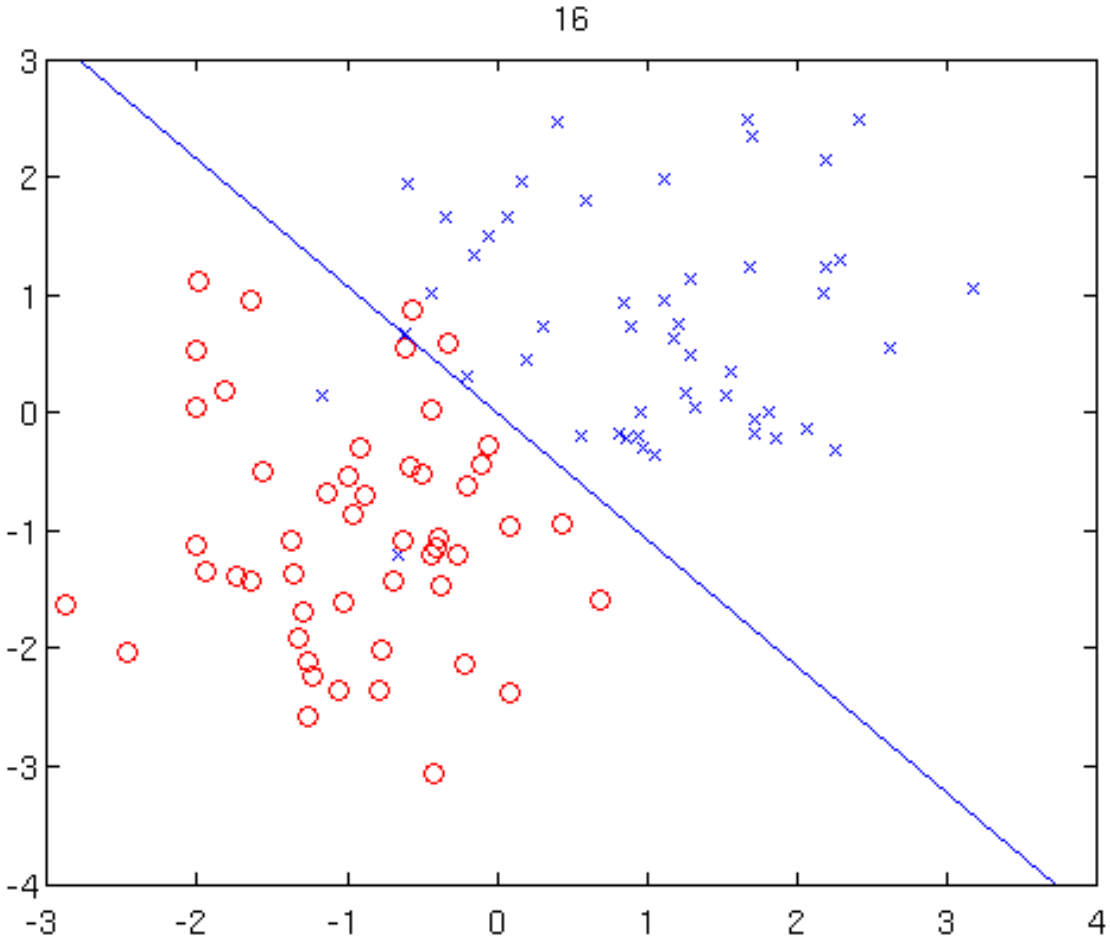

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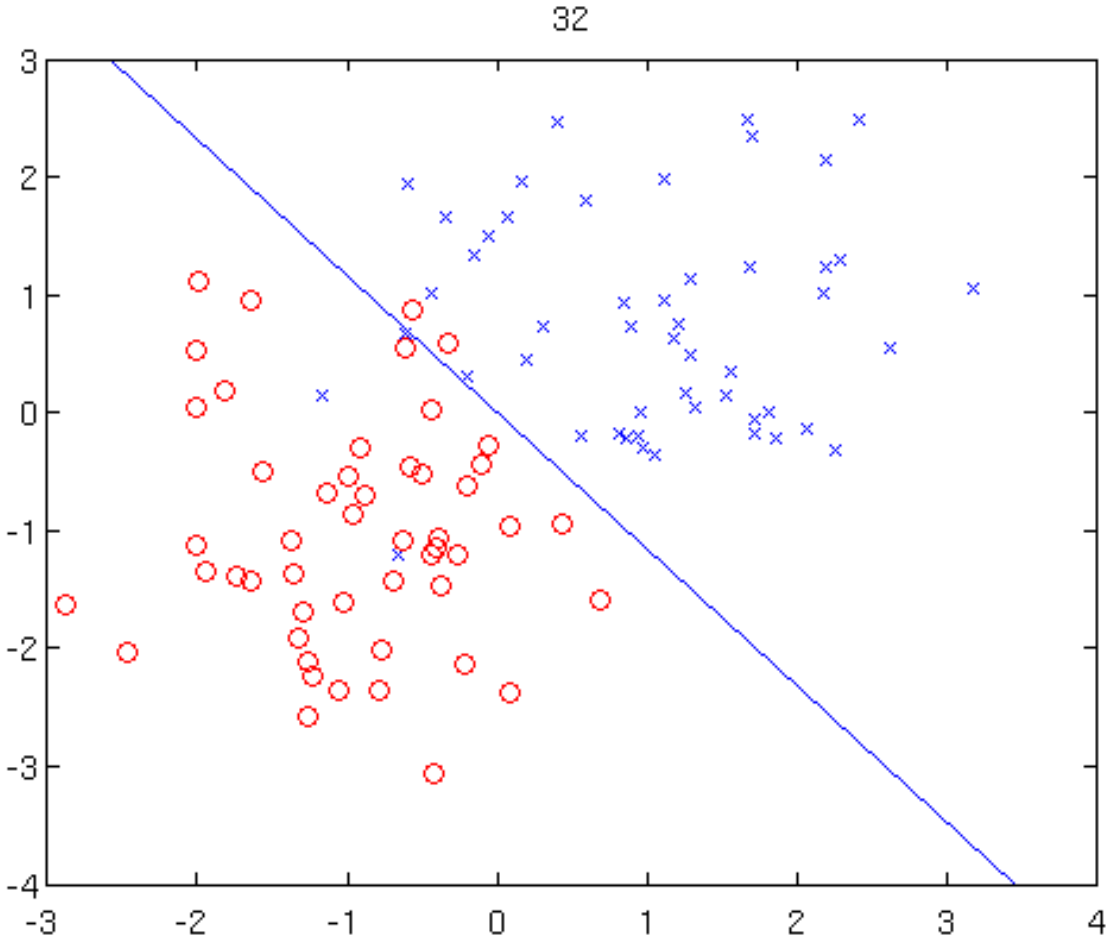

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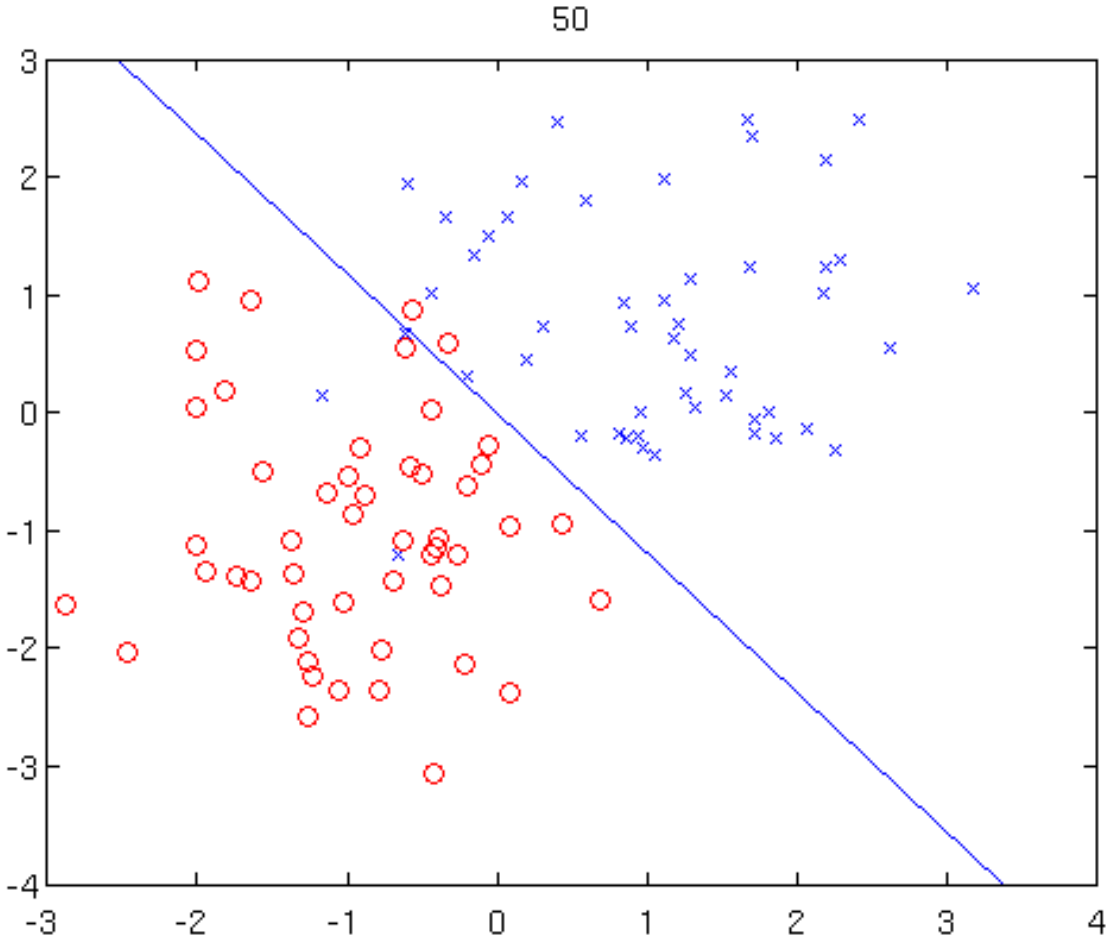


How does this get used in computer vision

- Step 1 - Define a labeling task
- Step 2 - Get a bunch of *training examples*, with *ground-truth labels*
- Step 3 - Train the classifier
- Step 4 - Get a bunch of *test examples*, also with *ground-truth labels*
- Step 5 - Test classifier on test examples

Let's talk more about step 3

- Each training example is numerically represented by a set of features.
- These features are stored in a vector. The vector for each training example will be denoted $\mathbf{x}_i$, where the i represents the ith training example.
- Each training example also has a label, l_i
- If we denote the weight vector as $\mathbf{w}$, then we can rewrite the log-loss criterion as

$$L = \sum_{i=1}^{N} \log(1 + \exp\left((-l_i(\mathbf{w}^T \mathbf{x}_i)\right)) \tag{1}$$

- (In the earlier example, $\mathbf{w} = [abc]$)

Is it really the same?

- Compare

$$L = \sum_{i=1}^{N} \log\left(1 + \exp(-l_i(ax_i + by_i + c))\right)$$

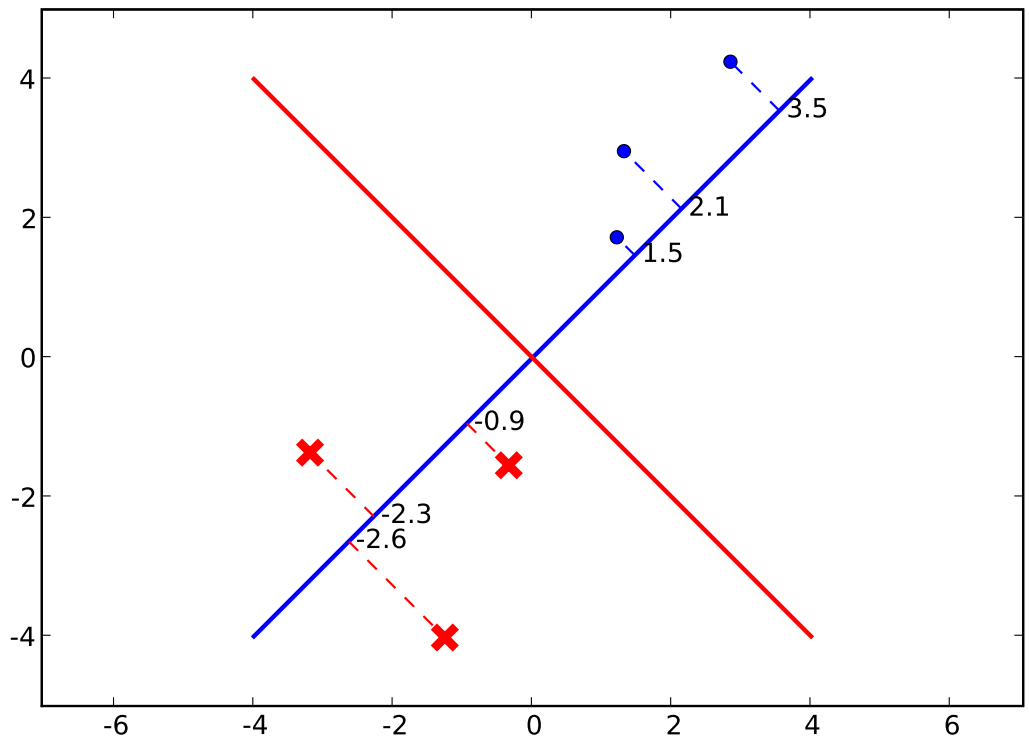
- With

$$L = \sum_{i=1}^{N} \log(1 + \exp\left((-l_i(\mathbf{w}^T\mathbf{x}_i)\right)) \tag{2}$$

- What's missing?
- In the upper equation c is a constant.
- This is called the bias term.

Function of the bias term

- Going back to this example:



How do we implement it?

- It's actually quite easy
- Make one feature be equal to all ones

How does this get used in computer vision

- Step 1 - Define a labeling task
- Step 2 - Get a bunch of *training examples*, with *ground-truth labels*
- Step 3 - Train the classifier
- Step 4 - Get a bunch of *test examples*, also with *ground-truth labels*
- Step 5 - Test classifier on test examples

Steps 4 and Steps 5 - Testing

- We want to want to know how well the system will perform on data that it has never seen
- *Overfitting* can be a problem, especially if there isn't much training data.
- In classification, we usually measure the percentage of points correctly labeled.